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Marie-Françoise Bidaut-Véron, Nguyen Anh Dao

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Abstract

Here we study the initial trace problem for the nonnegative solutions of equation

$$u_t - \Delta u + |\nabla u|^q = 0$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, where $q > 0$, and $\Omega = \mathbb{R}^N$, or Ω is a bounded domain of $\mathbb{R}^N$ and $u = 0$ on $\partial\Omega \times (0,T)$. We define the trace at $t = 0$ as a Borel measure $(\mathcal{S}, u_0)$, infinite on a closed set $\mathcal{S}$, where u_0 is a Radon measure on $\Omega\backslash\mathcal{S}$. We show that the trace is a Radon measure when $q \leqq 1$. We study the existence for $q \in (1,(N+2)/(N+1)$ and any given $(\mathcal{S}, u_0)$. When $\mathcal{S} = \overline{\omega} \cap \Omega$ (ω open$\subset \Omega$) existence is valid for $q \leqq 2$ when $u_0 \in L^1_{loc}(\Omega)$, for $q > 1$ when $u_0 = 0$. In particular there exists a self-similar nonradial solution with trace $(\mathbb{R}^{N+}, 0)$, with a growth rate of order $|x|^{q/(q-1)}$ as $|x| \to \infty$ for fixed t. Moreover the solutions with trace $(\overline{\omega}, 0)$ in $Q_{\mathbb{R}^N,T}$ may present a growth rate of order $t^{-1/(q-1)}$ in ω and of order $t^{-(2-q)/(q-1)}$ on $\partial\omega$.

Keywords Hamilton-Jacobi equation; Radon measures; Borel measures; initial trace; universal bounds

A.M.S. Subject Classification 35K15, 35K55, 35B33, 35B65, 35D30

Contents

1 Introduction

Here we consider the solutions of the parabolic Hamilton-Jacobi equation

$$u_t - \Delta u + |\nabla u|^q = 0 \tag{1.1}$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, $T \leqq \infty$, where $q > 0$, and $\Omega = \mathbb{R}^N$, or Ω is a smooth bounded domain of $\mathbb{R}^N$ and $u = 0$ on $\partial\Omega \times (0,T)$.

We mainly study the problem of initial trace of the *nonnegative* solutions. Our main questions are the following: Assuming that u is a nonnegative solution, what is the behaviour of u as t tends to 0? Does u converges to a Radon measure u_0 in Ω, or even to an unbounded Borel measure in Ω? Conversely, does there exist a solution with such a measure as initial data, and is it unique in some class?

In the sequel $\mathcal{M}(\Omega)$ is the set of Radon measures in $\Omega, \mathcal{M}_b(\Omega)$ the subset of bounded measures, and $\mathcal{M}^+(\Omega), \mathcal{M}_b^+(\Omega)$ are the cones of nonnegative ones. We say that a nonnegative solution u of (1.1) *has a trace* u_0 *in* $\mathcal{M}(\Omega)$ if $u(.,t)$ converges to u_0 in the weak* topology of measures:

$$\lim_{t\to 0} \int_\Omega u(.,t)\psi dx = \int_\Omega \psi du_0, \qquad \forall \psi \in C_c(\Omega). \tag{1.2}$$

First recall some known results. The Cauchy problem in $Q_{\mathbb{R}^N,T}$

$$(P_{\mathbb{R}^N,T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\mathbb{R}^N,T}, \\ u(x,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

and the Dirichlet problem in a bounded domain

$$(P_{\Omega,T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0. \end{cases} \tag{1.4}$$

have been the object of a rich literature, see among them [17], [3],[9], [7], [11], [27], [6], [13], [14], and references therein. The first studies of $(P_{\mathbb{R}^N,T})$ concern the existence of classical solutions, that means $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$, with smooth initial data: the case $u_0 \in C_b^2\left(\mathbb{R}^N\right)$ and $q > 1$, was studied in [3], and extended to any $u_0 \in C_b\left(\mathbb{R}^N\right)$ and $q > 0$ in [18]. Then the problem was studied in a semi-group formulation for rough initial data $u_0 \in L^r\left(\mathbb{R}^N\right)$, $r \geqq 1$, or $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$, [9], [11], [27], and in the larger class of weak solutions in [13], [14].

A critical value appears when $q > 1$:

$$q_* = \frac{N+2}{N+1}.$$

Indeed the problem with initial value $u_0 = \delta_0$, Dirac mass at 0 has a weak solution if and only if $q < q_*$, see [9], [13]. In the same range the problem has a unique very singular solution (in short V.S.S.) $Y_{\{0\}}$, such that

$$\lim_{t\to 0} \int_{|x|\geqq r} Y_{\{0\}}(.,t)dx = 0, \qquad \lim_{t\to 0} \int_{|x|<r} Y_{\{0\}}(.,t)dx = \infty, \qquad \forall r > 0,$$

see [26], [10], [8], [13]. It is radial and self-similar: $Y_{\{0\}}(x,t) = t^{-a/2}F(|x|/\sqrt{t})$, with

$$F \in C([0,\infty)), F(0) > 0, F'(0) = 0, \qquad \lim_{|\eta|\to\infty} e^{\frac{\eta^2}{4}} |\eta|^{N-a} F(\eta) = C > 0, \tag{1.5}$$

where

$$a = \frac{2-q}{q-1}. \tag{1.6}$$

It is clear that $Y_{\{0\}}$ does not admit a trace as a Radon measure. Otherwise, for any $q > 1$, the Dirichlet problem $(P_{\Omega,T})$ admits a solution U such that $\lim_{t\to 0}U(x,t) = \infty$ uniformly on the compact sets of Ω, see [17]. Thus we are lead to define an extended notion of trace.

The problem has been considered in [15], [23] for the semi-linear equation

$$u_t - \Delta u + u^q = 0, \tag{1.7}$$

with $q > 1$. Here another critical value $(N+2)/N$ is involved: there exist solutions with initial value δ_0 if and only if $q < (N+2)/N$, see [15], and then there exists a V.S.S., see [16], [19]. In [23] a precise description of the initial trace is given: any nonnegative solution admits a trace as an outer regular Borel measure $\mathcal{U}_0$ in Ω. Moreover if $q < (N+2)/N$, the problem is *well posed* in this set of measures in $\mathbb{R}^N$. The result of uniqueness lies on the monotony of the function $u \mapsto u^q$. If $q \geqq (N+2)/N$, necessary and sufficient conditions are given for existence, the problem admits a maximal solution, but uniqueness fails. Equation (1.7) admits a particular solution $((q-1)t)^{-1/(q-1)}$, which governs the upper estimates. Notice that the V.S.S. has precisely a behaviour in $t^{-1/(q-1)}$ at $x = 0$, as $t \to 0$.

Here we extend some of these results to equation (1.1). Compared to problem (1.7), new difficulties appear:

1) The first one concerns the a priori estimates. The equation (1.1) has no particular solution depending only on t. Note also that the sum of two supersolutions is not in general a supersolution. In [17] a universal upper estimate of the solutions u, of order $t^{-1/(q-1)}$, is proved for the Dirichlet problem. For the Cauchy problem, universal estimates of the gradient have been obtained for classical solutions with smooth data u_0, see [9], and [27]. They are improved in [12], where estimates of u of order $t^{-1/(q-1)}$ are obtained, see Theorem 2.9 below, and it is one of the key points in the sequel.

2) The second one comes from the fact that singular solutions may present two different levels of singularity as $t \to 0$. Notice that the V.S.S. $Y_{\{0\}}$ has a behaviour of order $t^{-a/2}$ smaller than $t^{-1/(q-1)}$.

3) The last one is due to the lack of monotony of the absorption term $|\nabla u|^q$. Thus many uniqueness problems are still open.

We first recall in Section 2 the notions of solutions, and precise the a priori upper and lower estimates, for the Cauchy problem or the Dirichlet problem. In Section 3 we describe the initial trace for $q > 1$:

Theorem 1.1 *Let $q > 1$. Let u be any nonnegative weak solution of (1.1) in any domain Ω. Then there exist a set $\mathcal{S} \subset \Omega$ such that $\mathcal{R} = \Omega \backslash \mathcal{S}$ is open, and a measure $u_0 \in \mathcal{M}^+(\mathcal{R})$, such that*

- *For any $\psi \in C_c^0(\mathcal{R})$,*

$$\lim_{t\to 0} \int_{\mathcal{R}} u(.,t)\psi = \int_{\mathcal{R}} \psi du_0. \tag{1.8}$$

- *For any $x_0 \in \mathcal{S}$ and any $\varepsilon > 0$*

$$\lim_{t\to 0} \int_{B(x_0,\varepsilon)\cap\Omega} u(.,t)dx = \infty. \tag{1.9}$$

The outer regular Borel measure $\mathcal{U}_0$ on Ω associated to the couple $(\mathcal{S}, u_0)$ defined by

$$\mathcal{U}_0(E) = \begin{cases} \int_E du_0 & \text{if } E \subset \mathcal{R}, \\ \infty & \text{if } E \cap \mathcal{S} \neq \emptyset, \end{cases}$$

is called the *initial trace* of u. The set $\mathcal{S}$ is called the set of *singular points* of $\mathcal{U}_0$ and $\mathcal{R}$ called the set of *regular points*, and u_0 the *regular part* of $\mathcal{U}_0$.

As $t \to 0$, we give lower estimates of the solutions on $\mathcal{S}$ of two types: of type $t^{-1/(q-1)}$ on $\overset{\circ}{\mathcal{S}}$ (if it is nonempty) and of type $t^{-a/2}$ on $\mathcal{S}$ (if $q < q_*$). Moreover we describe more precisely the trace for equation (1.1) in $Q_{\mathbb{R}^N,T}$:

Theorem 1.2 *Let $\mathcal{S}$ be closed set in $\mathbb{R}^N$, $\mathcal{S} \neq \mathbb{R}^N$, and $u_0 \in \mathcal{M}^+\left(\mathbb{R}^N \backslash \mathcal{S}\right)$. Let u be any nonnegative classical solution of (1.1) in $Q_{\mathbb{R}^N,T}$ (any weak solution if $q \leqq 2$), with initial trace $(\mathcal{S}, u_0)$.*

Then there exists a measure $\gamma \in \mathcal{M}^+(\mathbb{R}^N)$, concentrated on $\mathcal{S}$, such that $t^{1/(q-1)}u$ converges weak * to γ as $t \to 0$. And $\gamma \in L^\infty_{loc}(\mathbb{R}^N)$; in particular if $|\mathcal{S}| = 0$, then $\gamma = 0$; if $\mathcal{S}$ is compact, then $\gamma \in L^\infty(\mathbb{R}^N)$.

In Section 4 we study the existence and the behaviour of solutions with trace $(\overline{\omega} \cap \Omega, 0)$, where ω is a smooth open subset of Ω. We construct new solutions of (1.1) in $Q_{\mathbb{R}^N,T}$, in particular the following one:

Theorem 1.3 *Let $q > 1$, $q' = q/(q-1)$, and $\mathbb{R}^{N+} = \mathbb{R}^+ \times \mathbb{R}^{N-1}$. There exists a* ***nonradial*** *self-similar solution of (1.1) in $Q_{\mathbb{R}^N,T}$, with trace $(\overline{\mathbb{R}^{N+}}, 0)$, only depending on x_1 : $U(x,t) = t^{-a/2} f(t^{-1/2}x_1)$, where*

$$\lim_{\eta \to \infty} \eta^{-q'} f(\eta) = c_q = (q')^{-q'} (\frac{1}{q-1})^{\frac{1}{q-1}}, \qquad \lim_{\eta \to -\infty} e^{\frac{\eta^2}{4}} (-\eta)^{-\frac{3-2q}{q-1}} f(\eta) = C > 0.$$

Thus as $t \to 0$, $U(x,t)$ behaves like $t^{-1/(q-1)}$ for fixed $x \in \mathbb{R}^{N+}$, and $U(x,t) = f(0)t^{-a/2}$ for $x \in \partial\mathbb{R}^{N+}$. And for fixed $t > 0$, $U(x,t)$ is unbounded: it behaves like $x_1^{q'}$ as $x_1 \to \infty$.

By using U as a barrier, we can estimate precisely the two growth rates of the solutions in $Q_{\mathbb{R}^N,T}$ with trace $(\overline{\omega}, 0)$, on ω and on $\partial\omega$, for any $q > 1$, see Proposition 4.8.

In Section 5 we show the existence of solutions with initial trace $(\mathcal{S}, u_0)$, when $\mathcal{S} = \overline{\omega} \cap \Omega$ and $\omega \subset \Omega$ is open, and u_0 is a measure on $\Omega \backslash \overline{\omega}$, which can be unbounded, extending the results of [12, Theorem 1.4] relative to the case of a trace $(0, u_0)$:

Theorem 1.4 *Assume that $\Omega = \mathbb{R}^N$ (resp. Ω is bounded). Let ω be a smooth open subset of Ω, such that $\mathcal{R} = \Omega \backslash \overline{\omega}$ is nonempty, and let $\mathcal{S} = \overline{\omega} \cap \Omega$. Let $u_0 \in \mathcal{M}^+(\mathcal{R})$. We suppose that either $1 < q < q_*$, or $q_* \leqq q \leqq 2$ and $u_0 \in L^1_{loc}(\mathcal{R})$, or $q > 2$ and $u_0 \in L^1_{loc}(\mathcal{R})$ is limit of a nondecreasing sequence of continuous functions.*

Then there exists a weak solution u of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. a weak solution of $(D_{\Omega,T})$) such that u admits $(\mathcal{S}, u_0)$ as initial trace. Moreover as $t \to 0$, $u(.,t)$ converges to ∞ uniformly on any compact in ω, and uniformly on $\overline{\omega} \cap \Omega$ if $q < q_*$.

In the subcritical case $q < q_*$ we study the existence of solutions with trace $(\mathcal{S}, u_0)$ for any closed set $\mathcal{S}$ in Ω. Our main result is the following:

Theorem 1.5 *Let $1 < q < q_*$, and $\Omega = \mathbb{R}^N$ (resp. Ω is bounded). Let $\mathcal{S}$ be a closed set in $\mathbb{R}^N$, such that $\mathcal{R} = \mathbb{R}^N \backslash \mathcal{S}$ is nonempty. Let $u_0 \in \mathcal{M}^+(\mathcal{R})$.*

(i) Then there exists a minimal solution u of (1.1) with initial trace $(\mathcal{S}, u_0)$

(ii) If $\mathcal{S}$ is compact in Ω and $u_0 \in \mathcal{M}^+_b(\Omega)$ with support in $\mathcal{R} \cup \overline{\Omega}$, then there exists a maximal solution (resp. a maximal solution such that $u(.,t)$ converges weakly to u_0 in $\mathcal{R}$ as $t \to 0$).

In Section 6 we study equation (1.1) for $0 < q \leqq 1$, with more generally *signed* solutions, and the initial trace of the nonnegative ones. We first show the local regularity of the signed solutions, see Theorem 6.1. We prove a uniqueness result for the Dirichlet problem, extending to any $0 < q \leqq 1$ the results of [7], relative to the case $0 < q < 2/(N+1)$:

Theorem 1.6 *Let Ω be bounded, $0 < q \leqq 1$, and $u_0 \in \mathcal{M}_b(\Omega)$. Then there exists a unique weak (signed) solution u of problem $(P_{\Omega,T})$ with initial data u_0. Let $u_0, v_0 \in \mathcal{M}_b(\Omega)$ such that $u_0 \leqq v_0$. Then $u \leqq v$. In particular if $u_0 \geqq 0$, then $u \geqq 0$. If $u_0 \leqq 0$, then $u \leqq 0$.*

Finally we show that any nonnegative solution admits a trace as a Radon measure:

Theorem 1.7 *Let $0 < q \leqq 1$. Let u be any nonnegative weak solution of (1.1) in any domain Ω. Then u admits a trace u_0 in $\mathcal{M}^+(\Omega)$.*

2 First properties of the solutions

We set $Q_{\Omega,s,\tau} = \Omega \times (s,\tau)$, for any $0 \leqq s < \tau \leqq \infty$, thus $Q_{\Omega,T} = Q_{\Omega,0,T}$. We denote by $C(\Omega)$ the set of continuous functions in Ω, and $C_b(\Omega) = C(\Omega) \cap L^\infty(\Omega)$, $C_c(\Omega) = \{\varphi \in C(\Omega) : \mathrm{supp}\varphi \subset\subset \Omega\}$, and $C_0(\overline{\Omega}) = \left\{\varphi \in C(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega\right\}$.

Notation 2.1 *Let $\Omega = \mathbb{R}^N$ or Ω bounded, and $\Sigma \subset \Omega$. For any $\delta > 0$, we set*

$$\Sigma_\delta^{ext} = \{x \in \Omega : d(x,\Sigma) \leqq \delta\}, \qquad \Sigma_\delta^{int} = \{x \in \Sigma : d(x,\Omega\backslash\Sigma) > \delta\}. \tag{2.1}$$

2.1 Weak solutions and regularity

Definition 2.2 *Let $q > 0$ and Ω be any domain of $\mathbb{R}^N$. We say that a function u is a **weak solution** of equation of (1.1) in $Q_{\Omega,T}$, if $u \in C((0,T); L^1_{loc}(Q_{\Omega,T})) \cap L^1_{loc}((0,T); W^{1,1}_{loc}(\Omega))$, $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$, and u satisfies (1.1) in the distribution sense:*

$$\int_0^T \int_\Omega (-u\varphi_t - u\Delta\varphi + |\nabla u|^q \varphi)dxdt = 0, \quad \forall \varphi \in \mathcal{D}(Q_{\Omega,T}). \tag{2.2}$$

*We say that u is a **classical** solution of (1.1) in $Q_{\Omega,T}$ if $u \in C^{2,1}(Q_{\Omega,T})$ and satisfies(1.1) everywhere.*

For $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$, we say that u is a weak solution of $(P_{\mathbb{R}^N,T})$ if u is a weak solution of (1.1) with trace u_0.

Remark 2.3 *(i)If u is any nonnegative function such that $u \in L^1_{loc}(Q_{\Omega,T})$, and $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$, and satisfies (2.2), then u is a weak solution of (1.1). Indeed, since u is subcaloric, there holds $u \in L^\infty_{loc}(Q_{\Omega,T})), |\nabla u| \in L^2_{loc}(Q_{\Omega,T}))$, and $u \in C((0,T); L^\rho_{loc}(Q_{\Omega,T}))$, for any $\rho \geqq 1$, see [13, Lemma 2.4] for $q > 1$; the proof is still valid for any $q > 0$, since it only uses the fact that u is subcaloric.*

(ii) The weak solutions of $(P_{\mathbb{R}^N,T})$ are called weak $\mathcal{M}_{loc}$ solutions in [14].

Definition 2.4 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$. We say that a function u is a **weak solution** of*

$$(D_{\Omega,T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.3}$$

*if it is a weak solution of (1.1) such that $u \in C((0,T); L^1(\Omega))$, $u \in L^1_{loc}((0,T); W^{1,1}_0(\Omega))$, and $|\nabla u|^q \in L^1_{loc}((0,T); L^1(\Omega))$. We say that u is a **classical** solution of $(D_{\Omega,T})$ if $u \in C^{1,0}\left(\overline{\Omega} \times (0,T)\right)$ and u is a classical solution of (1.1).*

For $u_0 \in \mathcal{M}_b(\Omega)$, we say that u is a weak solution of $(P_{\Omega,T})$ if it is a weak solution of $(D_{\Omega,T})$ such that $u(.,t)$ converges weakly to u_0 in $\mathcal{M}_b(\Omega)$:

$$\lim_{t\to 0} \int_\Omega u(.,t)\psi dx = \int_\Omega \psi du_0, \qquad \forall \psi \in C_b(\overline{\Omega}). \tag{2.4}$$

Next we recall the regularity of the weak solutions for $q \leqq 2$, see [13, Theorem 2.9], [14, Corollary 5.14]:

Theorem 2.5 *Let $1 < q \leqq 2$.*

(i) Let Ω be any domain in $\mathbb{R}^N$, and u be a weak nonnegative solution of (1.1) in $Q_{\Omega,T}$. Then $u \in C^{2+\gamma,1+\gamma/2}_{loc}(Q_{\Omega,T})$ for some $\gamma \in (0,1)$. Thus for any sequence (u_n) of nonnegative weak solutions of (1.1) in $Q_{\Omega,T}$, uniformly locally bounded, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$.

(ii) Let Ω be bounded, and u be a weak nonnegative solution of $(D_{\Omega,T})$. Then $u \in C^{1,0}\left(\overline{\Omega} \times (0,T)\right)$ and $u \in C^{2+\gamma,1+\gamma/2}_{loc}(Q_{\Omega,T})$ for some $\gamma \in (0,1)$. For any sequence of weak nonnegative solutions (u_n) of $(D_{\Omega,T})$, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}_{loc}\left(\overline{\Omega} \times (0,T)\right)$ to a weak solution u of $(D_{\Omega,T})$.

2.2 Upper estimates

We first mention the universal estimates relative to classical solutions of the Dirichlet problem, see [17], and [13, Remark 2.8]:

Theorem 2.6 *Let $q > 1$, and Ω be any smooth bounded domain. and u be the classical solution of $(D_{\Omega,T})$ with initial data $u_0 \in C^{1,0}\left(\overline{\Omega}\right) \cap C_0\left(\overline{\Omega}\right)$. Then for any $t \in (0,T)$,*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq C(1+t^{-\frac{1}{q-1}})d(x,\partial\Omega), \qquad \|\nabla u(.,t)\|_{L^\infty(\Omega)} \leqq D(t), \tag{2.5}$$

where $C > 0$ and $D \in C((0,\infty))$ depend only of N, q, Ω. Thus, for any sequence (u_n) of classical solutions of $(D_{\Omega,T})$, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T})$ to a classical solution u of $(D_{\Omega,T})$.

Morever some local estimates of classical solutions have been obtained in [27], for any $q > 1$:

Theorem 2.7 *Let $q > 1$, and Ω be any domain in $\mathbb{R}^N$, and u be any classical solution of (1.1) in $Q_{\Omega,T}$. Then for any ball $B(x_0, 2\eta) \subset \Omega$, there holds in $Q_{B(x_0,\eta),T}$*

$$|\nabla u|(.,t) \leqq C(t^{-\frac{1}{q}} + \eta^{-1} + \eta^{-\frac{1}{q-1}})(1+u(.,t)), \qquad C = C(N,q). \tag{2.6}$$

Thus, for any sequence of classical solutions (u_n) of (1.1) in $Q_{\Omega,T}$, uniformly bounded in $L^\infty_{loc}(Q_{\Omega,T})$, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T})$ to a classical solution u of (1.1).

A local regularizing effect is proved in [12], easy consequence of the subcaloricity of the solutions:

Theorem 2.8 *Let $q > 1$. Let u be any nonnegative weak subsolution of (1.1) in $Q_{\Omega,T}$, and let $B(x_0, 2\eta) \subset \Omega$ such that u has a trace $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$. Then for any $\tau < T$, and any $t \in (0,\tau]$,*

$$\sup_{x \in B(x_0,\eta/2)} u(x,t) \leqq Ct^{-\frac{N}{2}}(t + \int_{B(x_0,\eta)} du_0), \qquad C = C(N,q,\eta,\tau). \tag{2.7}$$

Concerning the Cauchy problem in $Q_{\mathbb{R}^N,T}$, global regularizing effects have been obtained in [14] for weak solutions with trace u_0 in $L^r(\mathbb{R}^N), r \geqq 1$, or in $\mathcal{M}_b(\mathbb{R}^N)$. A universal estimate of the gradient was proved in [9] for any classical solution of (1.1) in $Q_{\mathbb{R}^N,\infty}$ such that $u \in C_b(\overline{Q_{\mathbb{R}^N,\infty}})$. From [12], this estimate is valid for any classical solution, implying growth estimates of the function:

Theorem 2.9 *Let $q > 1$. Let u be any classical solution, in particular any weak solution if $q \leqq 2$, of (1.1) in $Q_{\mathbb{R}^N,T}$. Then*

$$|\nabla u(.,t)|^q \leqq \frac{1}{q-1}\frac{u(.,t)}{t}, \qquad \text{in } Q_{\mathbb{R}^N,T}. \tag{2.8}$$

As a consequence, if there exists a ball $B(x_0, 2\eta)$ such that u has a trace $u_0 \in \mathcal{M}^+((B(x_0, 2\eta))$, then for any $t \in (0,T)$,

$$u(x,t) \leqq C(q)t^{-\frac{1}{q-1}}|x-x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \int_{B(x_0,\eta)} du_0), \qquad C = C(N,q,\eta). \tag{2.9}$$

Remark 2.10 *Estimate (2.9) is easy to obtain for classical solutions u such that $u \in C_b(\overline{Q_{\mathbb{R}^N,\infty}})$ or for limit a.e. of such functions, since it is a consequence of the universal gradient estimate, see [12, Theorem 4.1]. The difficult part of Theorem 2.9 is the extension of the gradient estimate without a priori estimates as $|x| \to \infty$.*

Finally we recall some well known estimates, useful in the subcritical case, see [4, Lemma 3.3]:

Theorem 2.11 *Let $q > 0$ and let Ω be any domain of $\mathbb{R}^N$ and u be any (signed) weak solution of equation of (1.1) in $Q_{\Omega,T}$ (resp. of $(D_{\Omega,T})$). Then, $u \in L^1_{loc}((0,T); W^{1,k}_{loc}(\Omega)$, for any $k \in [1, q_*)$, and for any open set $\omega \subset\subset \Omega$, and any $0 < s < \tau < T$,*

$$\|u\|_{L^k((s,\tau);W^{1,k}(\omega))} \leqq C(k,\omega)(\|u(.,s)\|_{L^1(\omega)} + \||\nabla u|^q + |\nabla u| + |u|\|_{L^1(Q_{\omega,s,\tau})}). \tag{2.10}$$

If Ω is bounded, any solution u of $(D_{\Omega,T})$ satisfies $u \in L^k((s,\tau); W^{1,k}_0(\Omega))$, for any $k \in [1, q_*)$, and

$$\|u\|_{L^k((s,\tau);W^{1,k}_0(\Omega))} \leqq C(k,\Omega)(\|u(.,s)\|_{L^1(\Omega)} + \||\nabla u|^q\|_{L^1(Q_{\Omega,s,\tau})}). \tag{2.11}$$

2.3 Uniqueness and comparison results

Next we recall some known results, for the Cauchy problem, see [11, Theorems 2.1,4.1,4.2 and Remark 2.1],[14, Theorem 2.3, 4.2, 4.25, Proposition 4.26], and for the Dirichlet problem, see [1, Theorems 3.1, 4.2], [7], [14, Proposition 5.17], [24].

Theorem 2.12 *Let $\Omega = \mathbb{R}^N$ (resp. Ω bounded). (i) Let $1 < q < q_*$, and $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$ (resp. $u_0 \in \mathcal{M}_b(\Omega)$). Then there exists a unique weak solution u of (1.1) with trace u_0 (resp. of $(P_{\Omega,T})$). If $v_0 \in \mathcal{M}_b(\Omega)$ and $u_0 \leqq v_0$, and v is the solution with trace v_0, then $u \leqq v$.*

(ii) Let $u_0 \in L^r(\Omega)$, $1 \leqq r \leqq \infty$. If $1 < q < (N + 2r)/(N + r)$, or if $q = 2$, $r < \infty$, there exists a unique weak solution u of $(P_{\mathbb{R}^N,T})$ (resp. $(P_{\Omega,T})$) such that $u \in C([0,T); L^r(\mathbb{R}^N))$. If $v_0 \in L^r(\mathbb{R}^N)$ and $u_0 \leqq v_0$, then $u \leqq v$. If u_0 is nonnegative, then for any $1 < q \leqq 2$, there still exists a weak nonnegative solution u of $(P_{\mathbb{R}^N,T})$ (resp. $(P_{\Omega,T})$) such that $u \in C([0,T); L^r(\mathbb{R}^N))$.

Remark 2.13 *Let $1 \leqq q < q_*$, and $u_0 \in \mathcal{M}^+_b(\mathbb{R}^N)$ and u be the solution of $(P_{\mathbb{R}^N,T})$ in $\mathbb{R}^N$, and u^Ω be the solution of $(D_{\Omega,T})$ for bounded Ω with initial data $u^\Omega_0 = u_0 \llcorner \Omega$, then $u^\Omega \leqq u$.*

We add to the results above a stability property needed below:

Proposition 2.14 *Assume that $1 < q < q_*$. Let $\Omega = \mathbb{R}^N$ (resp. Ω be bounded), and $u_{0,n}, u_0 \in \mathcal{M}^+_b(\Omega)$ such that $(u_{0,n})$ converge to u_0 weakly in $\mathcal{M}_b(\Omega)$. Let u_n, u be the (unique) nonnegative solutions of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$) with initial data $u_{0,n}, u_0$. Then (u_n) converges to u in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T})$ (resp. in $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0,T))$).*

Proof. (i) From [14, Theorem 2.2], (u_n) is uniformly locally bounded in $Q_{\mathbb{R}^N,T}$ in case $\Omega = \mathbb{R}^N$. From Theorem 2.5, one can extract a subsequence still denoted (u_n) converging in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T})$ (resp. $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0,T))$) to a classical solution w of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$). From uniqueness, we only have to show that $w(.,t)$ converges weakly in $\mathcal{M}_b(\Omega)$ to u_0. In any case, from [14, Theorem 4.15 and Lemma 5.11], $|\nabla u_n|^q \in L^1_{loc}([0,T); L^1(\Omega))$ and

$$\int_\Omega u_n(.,t)dx + \int_0^t \int_\Omega |\nabla u_n|^q dx \leqq \int_\Omega du_{0,n}, \tag{2.12}$$

and $\lim \int_\Omega du_{0,n} = \int_\Omega du_0$. Therefore (u_n) is bounded in $L^\infty((0,T), L^1(\Omega))$, and $(|\nabla u_n|^q)$ is bounded in $L^1_{loc}([0,T); L^1(\Omega))$. From Theorem 2.11, for any $k \in [1, q_*)$, (u_n) is bounded in $L^k((0,T), W^{1,k}_{loc}(\mathbb{R}^N))$ (resp. $L^k((0,T), W^{1,k}_0(\Omega))$. Then for any $\tau \in (0,T)$, ($|\nabla u_n|^q$) is equi-integrable in $Q_{B_R,\tau}$ for any $R > 0$ (resp. in $Q_{\Omega,\tau}$). For any $\xi \in C^1_c(\mathbb{R}^N)$ (resp. $\xi \in C^1_b(\Omega)$),

$$\int_\Omega u_n(.,t)\xi dx + \int_0^t \int_\Omega (\nabla u_n.\nabla\xi + |\nabla u_n|^q \xi) dxdt = \int_\Omega \xi du_{0,n},$$

and we can go to the limit and obtain

$$\int_{\Omega} w(.,t)\xi dx + \int_0^t \int_{\Omega} (\nabla w.\nabla \xi + |\nabla u|^q \xi) dxdt = \int_{\Omega} \xi du_0,$$

Then w is a weak solution of $(P_{\Omega,T})$, unique from Theorem 2.12, thus $w = u$. ■

Corollary 2.15 *Suppose $1 < q < q_*$, Ω bounded, and let $v \in C^{2,1}(Q_{\Omega,T}) \cap C^0(\overline{\Omega} \times (0,T))$ such that*

$$v_t - \Delta v + |\nabla v|^q \geqq 0, \qquad \text{in } \mathcal{D}'(Q_{\Omega,T}),$$

and $v_{/\Omega}$ has a trace $u_0 \in \mathcal{M}_b(\Omega)$. Let w be the solution of $(D_{\Omega,T})$ with trace u_0. Then $v \geqq w$.

Proof. Let $\epsilon > 0$ and (φ_n) be a sequence in $\mathcal{D}^+(\Omega)$ with values in $[0,1]$, such that $\varphi_n(x) = 1$ if $d(x,\partial\Omega) > 1/n$, and w_n^ϵ be the solution of $(D_{\Omega,T})$ with trace $\varphi_n v(.,\epsilon)$ at time 0, unique from Theorem 2.12. From [27, Proposition 2.1], $v(.,t+\epsilon) \geqq w_n^\epsilon$. As $n \to \infty$, $(\varphi_n v(.,\epsilon))$ converges to $v(.,\epsilon)$ in $L^1(\Omega)$, then from Proposition 2.14, (w_n^ϵ) converges to the solution w^ϵ of $(D_{\Omega,T})$ with trace $v(.,\epsilon)$. Then $v(.,t+\epsilon) \geqq w^\epsilon$. As $\epsilon \to 0$, $(v(.,\epsilon))$ converges to u_0 weakly in $\mathcal{M}_b(\Omega)$, thus (w^ϵ) converges to w, thus $v \geqq w$. ■

2.4 The case of zero initial data

Here we give more informations on the behaviour of the solutions with trace 0 on some open set. We show that the solutions are locally uniformly bounded on this set and converge locally exponentially to 0 as $t \to 0$, improving some results of [17] for the Dirichlet problem.

Lemma 2.16 *Let $\mathcal{F}$ be a closed set in $\mathbb{R}^N$, $\mathcal{F} \neq \mathbb{R}^N$ (resp. a compact set in Ω bounded).*

(i) Let u be a classical solution of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. $(D_{\Omega,T})$) such that $u \in C([0,T), C_b(\mathbb{R}^N))$ (resp. $u \in C([0,T); C_0(\overline{\Omega}))$ and $supp u(0) \subset \mathcal{F}$. Then for any $\delta > 0$, (resp. such that $\delta < d(\mathcal{F},\partial\Omega)/2$)

$$\|u(.,t)\|_{L^\infty(\Omega\backslash\mathcal{F}_\delta^{ext})} \leqq C(N,q,\delta)t, \qquad \forall t \in [0,T). \tag{2.13}$$

In particular $u(.,t)$ converges uniformly to 0 on $\Omega\backslash\mathcal{F}_\delta^{ext}$ as $t \to 0$. Moreover, there exist $C_{i,\delta} = C_{i,\delta}(N,q,\delta) > 0$ $(i = 1,2)$, and $\tau_\delta > 0$ such that

$$\|u(.,t)\|_{L^\infty(\Omega\backslash\mathcal{F}_\delta^{ext})} \leqq C_{1,\delta} e^{-\frac{C_{2,\delta}}{t}} \text{ on } (0,\tau_\delta]. \tag{2.14}$$

(ii) As a consequence, for any classical solution w of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. $(D_{\Omega,T})$) such that $w(.,t)$ converges to ∞ as $t \to 0$, uniformly on $\mathcal{F}_\delta^{ext}$, for some $\delta > 0$, there holds $u \leqq w$.

(iii) If $q \leqq 2$, then (i) still holds for any weak solution u of (1.1) (resp. of $(D_{\Omega,T})$) with trace 0 in $\mathcal{M}(\mathbb{R}^N\backslash\mathcal{F})$ (resp. which converges weakly to 0 in $\mathcal{M}_b(\Omega\backslash\mathcal{F})$), and (ii) holds if $\mathcal{F} \subset\subset \Omega$.

Proof. From [12, Lemma 3.2], for any domain Ω of $\mathbb{R}^N$, if u is any classical solution of (1.1) in $Q_{\Omega,T}$ such that $u \in C(\Omega \times [0,T))$, for any ball $B(x_0,3\eta) \subset \Omega$, and any $t \in [0,T)$,

$$\|u(.,t)\|_{L^\infty(B(x_0,\eta))} \leqq C(N,q)\eta^{-q'}t + \|u_0\|_{L^\infty(B(x_0,2\eta))}. \tag{2.15}$$

(i) Let Ω be arbitrary. For any $x_0 \in \Omega\backslash\mathcal{F}_\delta^{ext}$, taking $\eta = \delta/3$ we deduce (2.13) follows. Next suppose Ω bounded and $\mathcal{F}$ compact. Consider a regular domain Ω' such that $\mathcal{F}_{2\delta}^{ext} \subset \Omega' \subset\subset \Omega$. Let $\gamma = d(\overline{\Omega'},\partial\Omega)$. For any $x_0 \in \overline{\Omega'}\backslash\mathcal{F}_\delta^{ext}$, taking $\eta = \min(\delta/3,\gamma/3)$, we have $B(x_0,3\eta) \subset \Omega\backslash\mathcal{F}$ thus we still get (2.13). As a consequence $u(.,t) \leqq Ct$ in $\overline{\Omega'}\backslash\mathcal{F}_\delta^{ext}$, with $C = C(N,q,\delta,\gamma)$, in particular on $\partial\Omega'$. Following an argument of [13, Lemma 4.8], the function $z = u - Ct$ solves

$$z_t - \Delta z = -|\nabla u|^q - C \text{ in } \Omega\backslash\overline{\Omega'}$$

then z^+ is subcaloric and $z^+ = 0$ on the parabolic boundary of $\Omega\backslash\overline{\Omega'}$, thus $z^+ = 0$. Thus $u(.,t) \leqq Ct$ in $\overline{\Omega}\backslash\mathcal{F}_\delta^{ext}$.

Next consider the behaviour for small t. We use a supersolution in $B_1 \times [0,\infty)$ given in [25, Proposition 5.1]. Let $\alpha \in (0,1/2)$, and $d_\alpha(x)$ radial: $d_\alpha(x) = d_\alpha(|x|)$, with $d_\alpha \in C^2([0,1))$, $d_\alpha(r) = 1-r$ for $1-r < \alpha$, $d_\alpha(r) = 3\alpha/2$ for $1-r > 2\alpha$, $|\nabla d_\alpha| \leqq 1$, $|\Delta d_\alpha| \leqq C(N)d_\alpha^{-2}$. Let

$$v(x,t) = e^{\frac{1}{d_\alpha(x)} - m\frac{d_\alpha(x)^3}{t}}$$

with $m \leqq m(N)$ small enough. Then if $\alpha \leqq \alpha(N)$ small enough, there exists $\tau(\alpha) > 0$ such that v is a supersolution of (1.1) in $B_1 \times (0,\tau(\alpha)]$. Then $v(x,t) = C_1(\alpha)e^{-C_2(\alpha)/t}$ in $B_{1/2} \times (0,\tau(\alpha)]$. And v is infinite on $\partial B_1 \times (0,\tau(\alpha)]$ and vanishes on $B_1 \times \{0\}$. Then by scaling, for any $x_0 \in \mathbb{R}^N\backslash\mathcal{F}_\delta^{ext}$ (resp. $x_0 \in \overline{\Omega\backslash\mathcal{F}_\delta^{ext}}$), from the comparison principle in $B(x_0,\delta) \cap \overline{\Omega}$, we get

$$u(x_0,t) \leqq \delta^{-a}v(x_0/\delta, t/\delta^2) \leqq C_1(\alpha)\delta^{-a}e^{-C_2(\alpha)\delta^2/t} \tag{2.16}$$

and (2.14) follows.

(ii) Suppose that $w(.,t)$ converges to ∞ as $t \to 0$, uniformly on $\mathcal{F}_\delta^{ext}$. Then for any $\epsilon_0 > 0$, there exists $\tau_0 \in (0,T)$ such that $u(.,t) \leqq \epsilon_0$ in $\overline{\Omega}\backslash\mathcal{F}_\delta \times (0,\tau_0]$. Let $\epsilon < \tau_0$. then there exists $\tau_\epsilon < \tau_0$ such that for any $\theta \in (0,\tau_\epsilon)$, $w(.,\theta) \geqq \max_{\overline{\Omega}} u(.,\epsilon)$ in $\mathcal{F}_\delta$. Then $u(.,t+\epsilon) \leqq w(.,t+\theta) + \epsilon_0$, in $\overline{\Omega} \times (0,\tau_0 - \epsilon]$ from the comparison principle. As $\theta \to 0$, then $\epsilon \to 0$, we get $u(.,t) \leqq w(.,t) + \epsilon_0$, in $\overline{\Omega} \times (0,\tau_0]$. From the comparison principle, $u(.,t) \leqq w(.,t) + \epsilon_0$, in $\overline{\Omega} \times (0,T)$. As $\epsilon_0 \to 0$, we deduce that $u \leqq w$.

(iii) Assume $q \leqq 2$. First suppose $\Omega = \mathbb{R}^N$. From [13, Proposition 2.17 and Corollary 2.18], the extension $\overline{u}$ of u by 0 to $(-T,T)$ is a weak solution in $Q_{\mathbb{R}^N\backslash\mathcal{F},-T,T}$, hence $u \in C^{2,1}(\mathbb{R}^N\backslash\mathcal{F} \times [0,T))$, then u is a classical solution of (1.1) in $Q_{\mathbb{R}^N\backslash\mathcal{F},-T,T}$; thus (2.13) and (2.14) follow. Moreover, if $\mathcal{F}$ is compact, then $u(.,\epsilon/2) \in C_b(\mathbb{R}^N)$ from (2.13), then $u(.,\epsilon) \in C_b^2(\mathbb{R}^N)$, thus we still obtain $u \leqq w$ from the comparison principle. Next suppose Ω bounded and $\mathcal{F}$ compact. Arguing as in [13, Lemma 4.8], we show that $u \in C^0(\overline{\Omega\backslash\mathcal{F}_\delta^{ext}} \times [0,T))$, and $u(0) = 0$ in $\overline{\Omega\backslash\mathcal{F}_\delta^{ext}}$. We still get (2.13) by considering z as above, and using the Kato inequality, and (2.14) from the comparison principle. Moreover we still get $u \leqq w$. ∎

3 Existence of initial trace as a Borel measure

Recall a simple trace result of [13].

Lemma 3.1 *Let Ω be any domain of $\mathbb{R}^N$, and $U \in C((0,T);L^1_{loc}(\Omega))$ be any nonnegative weak solution of equation*

$$U_t - \Delta U = \Phi \quad \textit{in } Q_{\Omega,T}, \tag{3.1}$$

with $\Phi \in L^1_{loc}(Q_{\Omega,T})$, $\Phi \geqq -G$, where $G \in L^1_{loc}(\Omega \times [0,T))$. Then $U(.,t)$ admits a trace $U_0 \in \mathcal{M}^+(\Omega)$. Furthermore, $\Phi \in L^1_{loc}([0,T);L^1_{loc}(\Omega))$, and for any $\varphi \in C^2_c(\Omega \times [0,T))$,

$$-\int_0^T \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)dxdt = \int_\Omega \varphi(.,0)dU_0. \tag{3.2}$$

If Φ has a constant sign, then $U \in L^\infty_{loc}([0,T);L^1_{loc}(\Omega))$ if and only if $\Phi \in L^1_{loc}([0,T);L^1_{loc}(\Omega))$.

As a consequence, we get a characterization of the solutions of (1.1) in any domain Ω which have a trace in $\mathcal{M}^+(\Omega)$: as in [13, Proposition 2.15] in case $q > 1$, we find:

Proposition 3.2 *Let $q>0$. Let u be any nonnegative weak solution u of (1.1) in $Q_{\Omega,T}$. Then the following conditions are equivalent:*

(i) u has a trace u_0 in $\mathcal{M}^+(\Omega)$,

(ii) $u \in L^\infty_{loc}([0,T);L^1_{loc}(\Omega))$,

(iii) $|\nabla u|^q \in L^1_{loc}(\Omega \times [0,T))$.

And then for any $t \in (0,T)$, and any $\varphi \in C^1_c(\Omega \times [0,T))$,

$$\int_\Omega u(.,t)\varphi dx + \int_0^t \int_\Omega (-u\varphi_t + \nabla u.\nabla\varphi + |\nabla u|^q \varphi)dxdt = \int_\Omega \varphi(.,0)du_0. \tag{3.3}$$

And if $q>1$, for any nonnegative $\zeta, \xi \in C^1_c(\Omega)$,

$$\int_\Omega u(.,t)\zeta dx + \int_0^t \int_\Omega (\nabla u.\nabla\zeta + |\nabla u|^q \zeta)dxdt = \int_\Omega \zeta du_0, \tag{3.4}$$

$$\int_\Omega u(.,t)\xi^{q'}dx + \frac{1}{2}\int_0^t \int_\Omega |\nabla u|^q \xi^{q'}dx \leqq C(q)t\int_\Omega |\nabla\xi|^{q'}dx + \int_\Omega \xi^{q'}du_0. \tag{3.5}$$

Proof. The equivalence and equality (3.3) hold from Lemma 3.1. Moreoever for any $0<s<t<T$,

$$\begin{aligned}&\int_\Omega u(.,t)\xi^{q'}dx + \int_s^t \int_\Omega |\nabla u|^q \xi^{q'}dx = -q'\int_s^t\int_\Omega \xi^{1/(q-1)}\nabla u.\nabla\xi dx + \int_\Omega u(s,.)\xi^{q'}dx \\ &\leqq \frac{1}{2}\int_s^t \int_\Omega |\nabla u|^q \xi^{q'}dx + C(q)t\int_\Omega |\nabla\xi|^{q'}dx + \int_\Omega u(.,s)\xi^{q'}dx,\end{aligned}$$

hence we obtain (3.5) as $s \to 0$. ∎

Remark 3.3 *Note that $u \in L^\infty_{loc}([0,T);L^1_{loc}(\Omega))$ if and only if $\limsup_{t\to 0}\int_{B(x_0,\rho)} u(.,t)dx$ is finite, for any ball $B(x_0,\rho) \subset\subset \Omega$.*

Remark 3.4 *If Ω is bounded, $u_0 \in \mathcal{M}^+_b(\Omega)$ and u is any nonnegative classical solution (resp. weak solution if $q \leqq 2$) of $(P_{\Omega,T})$, then (3.5) still holds for any nonnegative $\xi \in C^1_b(\Omega)$. Indeed for any $0<s<t<T$, (3.4) is replaced by an inequality*

$$\int_\Omega u(.,t)\zeta dx + \int_0^t \int_\Omega (\nabla u.\nabla\zeta dx + |\nabla u|^q \zeta)dxdt = \int_0^t\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\zeta dsdt + \int_\Omega u(.,s)\zeta dx \leqq \int_\Omega u(.,s)\zeta dx,$$

and (3.5) follows as above.

Then we prove the trace Theorem:

Proof of Theorem 1.1. Let $q>1$. Let u be any nonnegative weak solution of (1.1) in $Q_{\Omega,T}$.

(i) Let $x_0 \in \Omega$. Then the following alternative holds (for any $\tau < T$):

$(A1)$ Either there exists a ball $B(x_0,\rho) \subset \Omega$ such that $\int_0^\tau \int_{B(x_0,\rho)} |\nabla u|^q dxdt < \infty$. Then from Lemma 3.1 in $B(x_0,\rho)$, there exists a measure $m_\rho \in \mathcal{M}^+(B(x_0,\rho))$, such that for any $\psi \in C^0_c(B(x_0,\rho))$,

$$\lim_{t\to 0}\int_{B(x_0,\rho)} u(.,t)\psi = \int_{B(x_0,\rho)} \psi dm_\rho, \tag{3.6}$$

$(A2)$ Or for any ball $B(x_0,\rho)\subset\Omega$ there holds $\int_0^\tau\int_{B(x_0,\rho)}|\nabla u|^q\,dxdt=\infty$. Taking $\psi=\xi^{q'}$with $\xi\in\mathcal{D}(\Omega)$, with $\xi\equiv 1$ on $B(x_0,\rho)$, with values in $[0,1]$, we have for any $0<t<\tau$,

$$\begin{aligned}\int_{B(x_0,\rho)}u(.,t)dx &\geqq \int_\Omega u(.,t)\xi^{q'}dx=\int_\Omega u(.,\tau)\xi^{q'}dx+\int_t^\tau\int_\Omega(q'\xi^{1/(q-1)}\nabla u.\nabla\xi+|\nabla u|^q\,\xi^{q'})dxdt\\ &\geqq\frac{1}{2}\int_t^\tau\int_\Omega|\nabla u|^q\,\xi^{q'})dxdt-C_q\int_t^\tau\int_\Omega|\nabla\xi|^{q'}\,dxdt,\end{aligned}$$

then

$$\lim_{t\to 0}\int_{B(x_0,\rho)}u(.,t)dx=\infty. \tag{3.7}$$

(ii) We define $\mathcal{R}$ as the open set of points $x_0\in\Omega$ satisfying $(A1)$ and $\mathcal{S}=\Omega\backslash\mathcal{R}$. Then from $(A1)$, there exists a unique measure $u_0\in\mathcal{M}(\mathcal{R})$ such that (1.8) holds; and (1.9) holds from $(A2)$. ■

3.1 First examples

1) Let $1<q<q_*$. (i) The V.S.S. $Y_{\{0\}}$ given by (1.5) in $Q_{\mathbb{R}^N,\infty}$ admits the trace $(\{0\},0)$.

(ii) Let Ω be bounded, and $x_0\in\Omega$. There exist a weak solution $Y^\Omega_{\{x_0\}}$of $(D_{\Omega,\infty})$ with trace $(\{x_0\},0))$, called V.S.S. in Ω relative to x_0. It is the unique weak solution such that

$$\lim_{t\to 0}\int_{B(x_0,\rho)}Y^\Omega_{\{x_0\}}(.,t)dx=\infty,\forall\rho>0,\qquad \lim_{t\to 0}\int_\Omega Y^\Omega_{\{x_0\}}(.,t)\psi dx=0,\forall\psi\in C_c(\overline{\Omega}\backslash\{x_0\}), \tag{3.8}$$

see [13, Theorem 1.5].

2) Let $1<q<q_*$. From [26], for any $\beta>F(0)$, where F is defined at (1.5), there exists a unique positive radial self-similar solution $U_\beta(x,t)=t^{-a/2}f_\beta(\frac{|x|}{\sqrt{t}})$ such that

$$f_\beta(0)=\beta,f'_\beta(0)=0,\quad\text{ and }\lim_{\eta\to\infty}f_\beta(\eta)\eta^a=C(\beta)>0; \tag{3.9}$$

then U_β has the trace $(\{0\},C(\beta)\,|x|^{-a})$. Notice that $x\mapsto|x|^{-a}$ belongs to $L^1_{loc}(\mathbb{R}^N\backslash\{0\})$ but not to $L^1_{loc}(\mathbb{R}^N)$.

3) Let $q_*<q<2$. For any $\beta>0$, there exists a unique solution as above, see [26]. Then U_β has the trace $(\emptyset,C(\beta)\,|x|^{-a})$; notice that $x\mapsto|x|^{-a}$ belongs to $L^1_{loc}(\mathbb{R}^N)$ but not to $L^1(\mathbb{R}^N)$.

4) Let Ω be bounded, and $q>1$. From [17], there exists a solution of $(D_{\Omega,\infty})$ which converges to ∞ uniformly on the compact sets of Ω as $t\to 0$. Then its trace is $(\Omega,0)$. See more details in Section 4.

3.2 Lower estimates

We first give *interior* lower estimates, valid for any $q>1$, by constructing a subsolution of the equation, with infinite trace in $B_{1/2}$ and compact support in B_1.

Proposition 3.5 *Let $q>1$, and Ω be any domain in $\mathbb{R}^N$, and let u be any classical solution u of (1.1) in $Q_{\Omega,T}$, such that u converges uniformly to ∞ on a ball $B(x_0,\rho)\subset\Omega$, as $t\to 0$. Then there exists $C=C(N,q,\rho)$ such that*

$$\liminf_{t\to 0}t^{\frac{1}{q-1}}u(x,t)\geqq C=C(N,q,\rho),\qquad\forall x\in B(x_0,\frac{\rho}{2}), \tag{3.10}$$

$$\liminf_{t\to 0}t^{\frac{1}{q-1}}u(x_0,t)\geqq C_q\rho^{q'},\qquad C_q=((q'(1+q'))^q(q-1))^{-\frac{1}{q-1}}. \tag{3.11}$$

Proof. Let $h, \lambda > 0$ be two parameters. We consider a function $t \in (0,\infty) \longmapsto \psi(t) = \psi_h(t) \in (1,\infty)$ depending on h, introduced in [12], solution of the ordinary differential equation

$$\psi_t + h(\psi^q - \psi) = 0 \quad \text{in } (0,\infty)\,, \qquad \psi(0) = \infty, \quad \psi(\infty) = 1, \tag{3.12}$$

given explicitely by $\psi(t) = (1 - e^{-h(q-1)t})^{-\frac{1}{q-1}}$; hence $\psi^q - \psi \geqq 0$, and $\psi(t) \geqq (h(q-1)t)^{-1/(q-1)}$ for any $t > 0$. Setting

$$V(x,t) = \psi(t) f(|x|), \qquad f(r) = (1 + q'r)(1-r)^{q'}, \qquad \forall r \in [0,1]\,,$$

we compute

$$D = V_t - \Delta V + |\nabla V|^q - \lambda V = (|f'|^q - hf)(\psi^q - \psi) + (|f'|^q - \Delta f - \lambda f)\psi.$$

Note that $f'(r) = -Mr(1-r)^{q'-1}$, with $M = q'(1+q')$. Thus $f'(0) = 0$ and f_0 is nonincreasing, and $|f'|^q - hf \leqq 0$ on $[0,1]$ for $h \geqq C_1 = M^q$. Otherwise $|f'|^q - \Delta f - \lambda f = (1-r)^{q'} J(r)$ with

$$J(r) = M^q r^q - \lambda(1 + q'r) + MG(r), \qquad G(r) = \frac{N - (N-1+q')r}{(1-r)^2}.$$

Then $J(0) = MN - \lambda\beta \leqq 0$ for $\lambda \geqq C_2 = NM$. We have

$$J'(r) = qM^q r^{q-1} - \lambda q' + MG'(r), \qquad G'(r) = (1-r)^{-3}(N + 1 - q' - (N-1+q')r).$$

If $q \leqq (N+1)/N$, there holds $q' > N+1$, hence $G' \leqq 0$, thus $J' \leqq 0$, for $\lambda \geqq (q-1)M^q$. If $q > (N+1)/N$, then $G'(r) \leqq 0$ for $r \geqq r_{N,q} = (N+1-q')/(N-1+q')$, and G' is continuous on $[0,1)$, hence bounded on $[0, r_{N,q}]$. Then $J' \leqq 0$ as soon as $\lambda \geqq C_3 = (q-1)M^q + (1+q')\max_{[0,r_{N,q}]} G'$. We fix $h = h(N,q) \geqq C_1$ and $\lambda = \lambda(N,q) \geqq \max(C_2, C_3)$, then $J(r) \leqq 0$ on $[0,1]$, thus $D \leqq 0$. Then the function

$$(x,t) \longmapsto w(x,t) = e^{-\lambda t} V(x,t) = e^{-\lambda t}\psi(t) f_0(|x|)$$

satisfies

$$w_t - \Delta w + e^{\lambda(q-1)t} |\nabla w|^q \leqq 0,$$

hence it is a subsolution of the Dirichlet problem $(D_{B_1,\infty})$, since $e^{\lambda(q-1)t} \geqq 1$. By scaling the function $(x,t) \longmapsto \tilde{w}(x,t) = \rho^{-a} w((x-x_0)/\rho, t/\rho^2)$ is a subsolution of $(D_{B(x_0,\rho),\infty})$. And u is a solution in $Q_{\Omega,T}$ which converges uniformly to ∞ on $B(x_0,\rho)$ as $t \to 0$. For given $\epsilon > 0$, there holds $\tilde{w}(.,\epsilon) \leqq m_\epsilon = \rho^{-a}\psi(\epsilon/\rho^2)$ in $B(x_0,\rho)$; and there exists $\tau_\epsilon \in (0,\epsilon)$ such that for any $\theta \in (0,\tau_\epsilon)$, $u(.,\theta) \geqq m_\epsilon$ in $B(x_0,\rho)$. Then $\tilde{w}(.,t+\epsilon) \leqq u(.,t+\theta)$ in $Q_{B(x_0,\rho),T-\epsilon}$. As $\theta \to 0$ and $\epsilon \to 0$, we get $\tilde{w} \leqq u$ in $Q_{B(x_0,\rho),T}$. And

$$\tilde{w}(x,t) \geqq \rho^{-a} e^{-\lambda t/\rho^2}\psi(t/\rho^2) \geqq (\rho/2)^{q'} e^{-\lambda t/\rho^2}(h(q-1)t)^{-1/(q-1)}$$

in $B(x_0,\rho/2)$, hence (3.10) holds. Taking $h = M^q = (q'(1+q'))^q$, there holds $u(x_0,t) \geqq \rho^{q'} e^{-\lambda t/\rho^2}(h(q-1)t)^{-1/(q-1)}$, thus (3.11) follows. ■

In case $1 < q < q_*$, we give a lower bound for all the weak solutions at any singular point, by an argument of stability-concentration, well-known for semilinear elliptic or parabolic equations, see [23].

Proposition 3.6 *Let $1 < q < q_*$. (i) Let u be any nonnegative weak solution u (1.1) in $Q_{\mathbb{R}^N,T}$ with singular set $\mathcal{S}$. Then for any $x_0 \in \mathcal{S}$, there holds $u(x,t) \geqq Y_{\{0\}}(x - x_0, t)$ in $Q_{\mathbb{R}^N,T}$, where $Y_{\{0\}}$ is the V.S.S. given at (1.5). In particular,*

$$u(x_0,t) \geqq C(N,q)t^{-a/2}, \qquad \forall t > 0. \tag{3.13}$$

(ii) Let Ω bounded, and u be any nonnegative weak solution u of $(D_{\Omega,T})$, with singular set $\mathcal{S}$. Then for any $x_0 \in \mathcal{S}$, $u(x,t) \geqq Y^{\Omega}_{\{x_0\}}(x,t)$ in $Q_{\Omega,T}$, where $Y^{\Omega}_{\{x_0\}}$ is given by (3.8). In particular,

$$\liminf_{t\to 0} t^{\frac{a}{2}} u(x_0,t) \geqq C(N,q) > 0. \tag{3.14}$$

In any case, $u(.,t)$ converges uniformly on $\mathcal{S}$ to ∞ as $t \to 0$.

Proof. (i) We can assume $x_0 = 0$. For any $\varepsilon > 0$, there holds $\lim_{t\to 0} \int_{B_\varepsilon} u(x,t)dx = \infty$. And $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$. We will prove that for fixed $k > 0$, there holds $u \geqq u^k$, where u^k is the unique solution in $\mathbb{R}^N$ with initial data $k\delta_0$, from Theorem 2.12. There exists $t_1 > 0$ such that $\int_{B_{2^{-1}}} u(x,t_1)dx > k$; thus there exists $s_{1,k} > 0$ such that $\int_{B_{2^{-1}}} \min(u(x,t_1), s_{1,k})dx = k$. By induction, there exists a decreasing sequence (t_n) converging to 0, and a sequence $(s_{n,k})$ such that $\int_{B_{2^{-n}}} \min(u(x,t_n), s_{n,k})dx = k$. Let $p \in \mathbb{N}$, $p > 1$. Denote by $u_{n,k,p}$ the solution of the Dirichlet problem $(D_{B_p,\infty})$, with initial data $u_{n,k,p}(.,0) = \min(u(.,t_n), s_{n,k})\chi_{B_{2^{-n}}}$. Then we get $u \geqq u_{n,k,p}$ in B_p, from Corollary 2.15. As $n \to \infty$, $(u_{n,k,p}(0))$ converges to $k\delta_0$ weakly in $\mathcal{M}_b(B_p)$. Indeed for any $\psi \in C^+(\overline{B_p})$,

$$\left| \int_{B_p} u_{n,k,p}(0)\psi dx - k\psi(0) \right| = \left| \int_{B_{2^{-n}}} \min(u(x,t_n), s_{n,k})\psi dx - k\psi(0) \right| \leqq k \left\| \psi - \psi(0) \right\|_{L^\infty(B_{2^{-n}})}.$$

Then $(u_{n,k,p})$ converges in $C^{2,1}_{loc}(Q_{B_p,T})$ to the solution u^{k,B_p} of the problem in B_p with initial data $k\delta_0$, from Proposition 2.14. Thus $u \geqq u^{k,B_p}$. Finally, as $p \to \infty$, u^{k,B_p} converges to u^k from [13, Lemma 4.6] and uniqueness of u^k; thus $u \geqq u^k$. As $k \to \infty$, (u^k) converges to $Y_{\{0\}}$, hence $v \geqq Y_{\{0\}}$. Then (3.13) holds with $C = F(0)$ given by (1.5).

(ii) In the same way, denote by $u^{k,\Omega}_{x_0}$, u_{n,k,x_0} the solutions of the Dirichlet problem $(D_{\Omega,\infty})$, with respective initial data $k\delta_{x_0}$and $T_{s_{n,k}}v(.,t_n)\chi_{B_{(x_0,2^{-n}d)}}$, where $d = d(x_0, \partial\Omega)$. Then as above we get $u \geqq u_{n,k,x_0}$ in Ω, then $u \geqq u^{k,\Omega}_{x_0}$. As $k \to \infty$, $(u^{k,\Omega})$ converges to $Y^{\Omega}_{\{x_0\}}$, and moreover, for any $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon, d)$ such that $Y_{\{x_0\}}(x,t) = Y(x - x_0, t) \leqq Y^{\Omega}_{\{x_0\}} + \varepsilon$ in $\overline{\Omega} \times (0,\tau)$, see the proof of [13, Theorem 1.5]. Then $u \geqq Y^{\Omega}_{\{x_0\}}$ and (3.14) follows by taking $\varepsilon = F(0)/2$. ■

Remark 3.7 *As a consequence, for $1 < q < q_*$, there exists no weak solution u of (1.1) in $Q_{\mathbb{R}^N,T}$ with singular set $\mathcal{S} = \mathbb{R}^N$. Indeed if u exists, u satisfies (3.13), then u converges uniformly on $\mathbb{R}^N$ as $t \to 0$. Then for any $k > 0$ and any $\varphi \in \mathcal{D}^+(B_1)$, $\varphi = 1$ in $B_{1/2}$, u is greater than the solution $u_{k,p}$ with initial trace $k\varphi(x/p)$. As $p \to \infty$, $u_{k,p}$ tends to the unique solution u_k with initial data k, namely $u_k \equiv k$, thus $u \geqq k$ for any $k > 0$, which is contradictory. The question is open for $q \geqq q_*$.*

Remark 3.8 *Another question is to know for which kind of solutions (3.14) still holds when $q \geqq q_*$. We give a partial answer in Section 4, see Proposition 4.8.*

3.3 Trace of the Cauchy problem

In this part we show Theorem 1.2, based on the universal estimate of Theorem 2.9.

Proof of Theorem 1.2. (i) From Theorem 2.9, u satisfies (2.8). Reporting in (1.1), we deduce

$$u_t - \Delta u + \frac{1}{q-1}\frac{u}{t} \geqq 0.$$

Setting $y = t^{1/(q-1)}u$, we get that

$$y_t - \Delta y = t^{\frac{1}{q-1}}\left(\frac{1}{q-1}\frac{u}{t} - |\nabla u|^q\right) \geqq 0$$

in $Q_{\mathbb{R}^N,T}$, thus y has a trace $\gamma \in \mathcal{M}^+(\mathbb{R}^N)$, see Lemma 3.1. Since $u(.,t)$ converges weak* to u_0 in $\mathbb{R}^N \backslash \mathcal{S}$, we find that supp $\gamma \subset \mathcal{S}$. Let $B(x_0, 2\eta) \subset \mathbb{R}^N \backslash \mathcal{S}$. From (2.9), we have

$$y(x,t) \leqq C(q)\left|x - x_0\right|^{q'} + C(1 + t^{q'} + t^{\frac{1}{q-1}} \int_{B(x_0,\eta)} du_0), \qquad C = C(N,q,\eta),$$

hence $\gamma \in L^\infty_{loc}(\mathbb{R}^N)$. ■

Remark 3.9 *In particular for $q < q_*$, the V.S.S. $Y_{\{0\}}$ in $\mathbb{R}^N$ satisfies $\gamma = 0$, which can be checked directly, since $lim_{t\to 0} t^{1/(q-1)-a/2} = 0$. The function U given at Theorem 1.3 satisfies $\gamma(x) = c_q(x_1^+)^{q'}$.*

4 Solutions with trace $(\overline{\omega} \cap \Omega, 0)$, ω open

Here we extend and improve the pionneer result of [17], valid for the Dirichlet problem in Ω bounded. In case of the Cauchy problem, the estimates (2.6) and (2.9) are essential for existence.

Theorem 4.1 *Let $q > 1$ and ω be a smooth open set in $\mathbb{R}^N$ with $\omega \neq \mathbb{R}^N$ (resp. a smooth open set in Ω bounded). There exists a classical solution $u = Y_{\overline{\omega}}$ (resp. $u = Y_{\overline{\omega}}^{\Omega}$) of (1.1) in $Q_{\mathbb{R}^N,\infty}$ (resp. of $(D_{\Omega,\infty})$), with trace $(\overline{\omega} \cap \Omega, 0)$. Moreover it satisfies uniform properties of convergence:*

$$\lim_{t\to 0} \inf_{x\in K} u(x,t) = \infty \quad \forall K \text{ compact } \subset \omega, \qquad \lim_{t\to 0} \sup_{x\in K} u(x,t) = 0 \quad \forall K \text{ compact} \subset \overline{\Omega}\backslash\overline{\omega}. \tag{4.1}$$

More precisely, for any $\delta > 0$,

$$\liminf_{t\to 0} t^{\frac{1}{q-1}} u(x,t) \geqq C(N,q)\delta^{q'}, \qquad \forall x \in \overline{\omega}_\delta^{int}, \tag{4.2}$$

$$\sup_{\Omega\backslash\overline{\omega}_\delta^{ext}} u(x,t) \leqq C(N,q,\delta)t, \qquad \forall t > 0. \tag{4.3}$$

If $q < q_*$, then for any $x \in \overline{\omega} \cap \Omega$,

$$\inf_{t>0} t^{\frac{a}{2}} u(x,t) \geqq C(N,q) > 0 \qquad (\text{resp. } \liminf_{t\to 0} t^{\frac{a}{2}} u(x,t) \geqq C(N,q) > 0). \tag{4.4}$$

Moreover, if $\Omega = \mathbb{R}^N$, the function $Y_{\overline{\omega}}$ satisfies the growth condition in $Q_{\mathbb{R}^N,\infty}$

$$Y_{\overline{\omega}}(x,t) \leqq C(t + t^{-\frac{1}{q-1}})(1 + |x|^{q'}), \qquad C = C(N,q,\omega) \tag{4.5}$$

Proof. First suppose Ω bounded, then $\overline{\omega}$ is a compact set in $\mathbb{R}^N$. We consider a nondecreasing sequence (φ_p) of nonnegative functions in $C_c^1(\Omega)$, with support in $\overline{\omega}$, such that $\varphi_p \geqq p$ in $\overline{\omega}_{1/p}^{int}$, and the nondecreasing sequence of classical solutions u_p^{Ω} with initial data φ_p. From Theorem 2.6, (u_p^{Ω}) converges in $C_{loc}^{2,1}(Q_{\Omega,T})$ to a solution $Y_{\overline{\omega}}^{\Omega}$ of $(D_{\Omega,T})$. Then by construction of u_p^{Ω}, $Y_{\overline{\omega}}^{\Omega}(.,t)$ converges uniformly to ∞ on the compact sets in ω. Then the conclusions hold from Lemma 2.16, Propositions 3.5 and 3.6.

Next suppose $\Omega = \mathbb{R}^N$. We can construct a nondecreasing sequence $(\varphi_p)_{p>p_0}$ of functions in $C_b^+(\mathbb{R}^N)$, with support in $\overline{\omega} \cap \overline{B_p}$, such that $\varphi_p \geqq p$ on $\overline{\omega}_{1/p}^{int} \cap B_{p-1/p}$. Let u_p be the classical solution of (1.1) in $Q_{\mathbb{R}^N,\infty}$ with initial data φ_p. Since $\omega \neq \mathbb{R}^N$, there exists a ball $B(x_0,\eta) \subset \mathbb{R}^N\backslash\overline{\omega})$. From (2.9),

$$u_p(x,t) \leqq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(N,q,\eta)(t^{-\frac{1}{q-1}} + t), \tag{4.6}$$

thus (u_p) is locally uniformly bounded in $Q_{\mathbb{R}^N,\infty}$. From Theorem 2.7, (u_p) converges in $C_{loc}^{2,1}(Q_{\mathbb{R}^N,\infty})$ to a classical solution $Y_{\overline{\omega}}$ of (1.1) in $Q_{\mathbb{R}^N,\infty}$. Then by construction of u_p, $Y_{\overline{\omega}}(.,t)$ converges uniformly to ∞ on the compact sets in ω, and the conclusions follow as above. Moreover, from (4.6), $Y_{\overline{\omega}}$ satisfies (4.5). ■

Remark 4.2 *Moreover, from the construction of the solutions, denoting by y_φ the solution of (1.1) with initial data $\varphi \in C_b^+(\mathbb{R}^N) \cap C_0^+(\overline{\omega})$ (resp. the solution of $(D_{\Omega,\infty})$ with initial data $\varphi \in C_0^+(\overline{\Omega})$) we get the relations*

$$Y_{\overline{\omega}}^{\Omega} = \sup_{\varphi\in C_0^+(\overline{\Omega}), supp\varphi\subset\overline{\omega}} y_\varphi, \qquad Y_{\overline{\omega}} = \sup_{\varphi\in C_b^+(\mathbb{R}^N), supp\varphi\subset\overline{\omega}} y_\varphi. \tag{4.7}$$

Indeed we get $y_\varphi \leqq Y_{\overline{\omega}}$, for any nonnegative $\varphi \in C_c^1(\mathbb{R}^N)$ (resp. $C_c^1(\Omega)$) with $supp\varphi \subset \overline{\omega}$, and the relation extends to any $\varphi \in C_b^+(\mathbb{R}^N)$ (resp. $C_0^+(\overline{\Omega})$), from uniqueness of y_φ.

Remark 4.3 *When Ω is bounded, and $\overline{\omega} \subset \Omega$, or $\omega = \Omega$, it was shown in [17] that there exists a solution $Y_{\overline{\omega}}^{\Omega}$ satisfying (4.1). Moreover, using the change of unknown $v = e^{-u}$, they proved that if $\omega \subset\subset \Omega$, then for any $x \in \partial\omega$,*

$$\lim_{t\to 0} Y_{\overline{\omega}}^{\Omega}(x,t) = \infty, \text{ if } q < 2; \quad \lim_{t\to 0} Y_{\overline{\omega}}^{\Omega}(x,t) = \ln 2, \text{ if } q = 2; \quad \lim_{t\to 0} Y_{\overline{\omega}}^{\Omega}(x,t) = 0, \qquad \text{if } q > 2. \tag{4.8}$$

Next we study the question of the *uniqueness* of solutions with trace $(\overline{\omega}, 0)$ which appears to be delicate. A first point is to precise in what class of solutions the uniqueness may hold, in particular in what sense the initial data are achieved.

Definition 4.4 *Let $\Omega = \mathbb{R}^N$ (resp. Ω bounded) and ω be a open set in Ω. We denote by $\mathcal{C}$ the class of classical solutions of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$) satisfying (4.1). We denote by $\mathcal{W}$ the class of weak solutions of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$) with trace $(\overline{\omega}, 0)$.*

In [17], the authors consider the class $\mathcal{C}$. They show that if $\overline{\omega}$ is compact contained in Ω bounded and ω, Ω are starshaped with respect to the same point or $q \geqq 2$, then $Y_{\overline{\omega}}^{\Omega}$ is unique in that class. But we cannot ensure that *any weak solution* u with trace $(\overline{\omega}, 0)$ converges *uniformly* to ∞ on the compact sets in ω. And in case $q > 2$ we even do not know if u is continuous. Here we give some partial results, where we do not suppose that Ω is starshaped.

Theorem 4.5 *(i) Let $q > 1$. Under the assumptions of Theorem 4.1, $Y_{\overline{\omega}} = \sup Y_{\overline{\omega}_\delta^{int}}$ and $Y_{\overline{\omega}}$ is a minimal solution in the class $\mathcal{C}$ (resp. $Y_{\overline{\omega}}^{\Omega} = \sup Y_{\overline{\omega}_\delta^{int}}^{\Omega}$ and $Y_{\overline{\omega}}^{\Omega}$ is a minimal solution in the class $\mathcal{C}$). If ω is compact, $\overline{u}_{\overline{\omega}} = \inf_{\delta>0} Y_{\overline{\omega}_\delta^{ext}}$ is a maximal solution of (1.1) in $Q_{\mathbb{R}^N,T}$ in the class $\mathcal{C}$ (resp. if $\omega \subset\subset \Omega$, then $\overline{u}_{\overline{\omega}}^{\Omega} = \inf_\delta Y_{\overline{\omega}_\delta^{ext}}^{\Omega}$ is a maximal solution of $(D_{\Omega,T})$ in the class $\mathcal{C}$).*

(ii) Let $1 < q \leqq 2$ and suppose ω compact (resp. $\omega \subset\subset \Omega$). Then the function $\overline{u}_{\overline{\omega}}$(resp. $\overline{u}_{\overline{\omega}}^{\Omega}$) defined above is maximal in the class $\mathcal{W}$. If ω is starshaped, then $Y_{\overline{\omega}}$ (resp. $Y_{\overline{\omega}}^{\Omega}$) is the unique solution of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$) in the class $\mathcal{C}$.

(iii) Let $1 < q < q_*$ and suppose ω compact (resp. $\omega \subset\subset \Omega$). Then $\mathcal{W} = \mathcal{C}$. Thus $Y_{\overline{\omega}}$ (resp. $Y_{\overline{\omega}}^{\Omega}$) is minimal in the class $\mathcal{W}$. If ω is starshaped it is unique in the class $\mathcal{W}$.

Proof. (i) Let Ω be bounded. Let v be any classical solution of $(D_{\Omega,T})$ satisfying (4.1). Let $\varphi \in C_0^+(\overline{\Omega})$ with supp$\varphi \subset \overline{\omega}$. Then there exists a nondecreasing sequence $(\varphi_n) \in C_0^+(\overline{\Omega})$, with support in $\overline{\omega}_{1/n}^{int}$, converging to φ in $C^+(\overline{\Omega})$. Then the function $(y_{\varphi_n}(.,t)))$ defined at Remark 4.2 converges to $y_\varphi(.,t)$ in $C(\overline{\Omega})$, uniformly for $t > 0$. For fixed n, let $\epsilon \in (0,T)$. Since $v(.,t)$ converges uniformly to ∞ on the compact sets of $\overline{\omega}$, and $\varphi_n = 0$ in $\overline{\Omega} \backslash \overline{\omega}_{1/n}^{int}$, there exists $\theta_n \in (0,\epsilon)$ such that $\inf v(.,t) \geqq \max \varphi_n \geqq \max y_{\varphi_n}(.,\epsilon)$ for any $t \leqq \theta_n$. Then $v \geqq y_{\varphi_n}$ on $[\epsilon, T)$ from the comparison principle, hence $v \geqq y_\varphi$. Then $Y_{\overline{\omega}}^{\Omega}$ is minimal in the class $\mathcal{C}$. Moreover for any $\delta > 0$, $Y_{\overline{\omega}_\delta^{int}}^{\Omega} \leqq Y_{\overline{\omega}}^{\Omega}$, and from (4.7),

$$\sup Y_{\overline{\omega}_\delta^{int}}^{\Omega} = \sup_\delta \Big(\sup_{\varphi \in C_0^+(\overline{\Omega}), \mathrm{supp}\varphi \subset \overline{\omega}_\delta^{int}} y_\varphi \Big) = \sup_{\varphi \in C_0^+(\overline{\Omega}), \mathrm{supp}\varphi \subset \overline{\omega}} y_\varphi = Y_{\overline{\omega}}^{\Omega}.$$

Now consider the case $\Omega = \mathbb{R}^N$. Let v be any classical solution in $Q_{\mathbb{R}^N,T}$ satisfying (4.1). Let $\varphi \in C_c^+(\mathbb{R}^N)$, with supp$\varphi \subset \overline{\omega}$. As above we deduce that $v \geqq y_\varphi$. From the uniqueness of the solutions, we deduce that $v \geqq y_\varphi$, for any $\varphi \in C_b^+(\mathbb{R}^N)$, with supp$\varphi \subset \overline{\omega}$. Then $Y_{\overline{\omega}}$ is minimal in the class $\mathcal{C}$. As above we obtain $Y_{\overline{\omega}} = \sup Y_{\overline{\omega}_\delta^{int}}$.

Assume that $\Omega = \mathbb{R}^N$ and $\overline{\omega}$ is compact. For $\delta > 0$ we consider the function $Y_{\overline{\omega}_\delta^{ext}}$ constructed as above. Then by construction, $Y_{\overline{\omega}} \leqq Y_{\overline{\omega}_\delta^{ext}}$. Taking $\delta_n \to 0$, $(Y_{\overline{\omega}_{\delta_n}^{ext}})$ decreases to a classical solution $\overline{u}_{\overline{\omega}}$ of (1.1) in $Q_{\mathbb{R}^N,T}$ from Theorem 2.7 thus $\overline{u}_{\overline{\omega}} \geqq Y_{\overline{\omega}}$, then $\overline{u}_{\overline{\omega}}$ satisfies (4.1). Moreover let v be any solution in the class $\mathcal{C}$.

From Lemma 2.16 (ii), $v \leqq Y_{\overline{\omega}^{ext}_{\delta}}$, then $v \leqq \overline{u}_{\overline{\omega}}$, thus $\overline{u}_{\overline{\omega}}$ is maximal. Next assume Ω bounded and $\omega \subset\subset \Omega$; the result follows as above by taking $\delta < \delta_0$ small enough such that $\overline{\omega}^{ext}_{\delta_0} \subset \Omega$ and using Theorem 2.6.

(ii) For $q \leqq 2$, $\overline{u}_{\overline{\omega}}$(resp. $\overline{u}^{\Omega}_{\overline{\omega}}$) is also maximal in the class $\mathcal{W}$, from Lemma 2.16 (iii). But we cannot ensure that is minimal in this class.

Suppose that ω is starshaped, then $Y_{\overline{\omega}}(x,t) = k^a Y_{k\overline{\omega}}(kx, k^2t)$, from (4.7). As above, any weak solution v of (1.1) in $Q_{\mathbb{R}^N,T}$ with trace $(\overline{\omega}, 0)$ satisfies $v \leqq Y_{k\overline{\omega}}$ for any $k > 1$, hence $v \leqq Y_{\overline{\omega}}$ as $k \to 1$, thus $\overline{u}_{\overline{\omega}} \leqq Y_{\overline{\omega}}$, hence $\overline{u}_{\overline{\omega}} = Y_{\overline{\omega}}$. We get uniqueness in the class $\mathcal{C}$. Now any weak solution w of $(D_{\Omega,T})$ with trace $(\overline{\omega}, 0)$ also satisfies $w \leqq Y_{k\overline{\omega}}$ in $\overline{\Omega} \times (0,T)$ for any $k > 1$, then also $Y^{\Omega}_{\overline{\omega}} \leqq \overline{u}^{\Omega}_{\omega} \leqq Y_{k\overline{\omega}}$. Thus as $k \to 1$, one gets $Y^{\Omega}_{\overline{\omega}} \leqq \overline{u}^{\Omega}_{\omega} \leqq Y_{\overline{\omega}}$. Let $\epsilon_0 > 0$. We fix $\delta > 0$ such that $\overline{\omega}^{ext}_{\delta} \subset \Omega$. From Lemma 2.16 (i), we get $Y_{\overline{\omega}}(.,t) \leqq C(N,q,\delta)t$ on $\partial\Omega$; hence there exists $\tau_0 > 0$ such that $Y_{\overline{\omega}} \leqq \epsilon_0$ on $\partial\Omega \times (0, \tau_0]$; thus, for any $\eta < 1$, $Y_{\eta\overline{\omega}} \leqq Y^{\Omega}_{\overline{\omega}} + \epsilon_0$, in $\overline{\Omega} \times (0, \tau_0]$. As $\eta \to 1$ we get $Y_{\overline{\omega}} \leqq Y^{\Omega}_{\overline{\omega}} + \epsilon_0$, in $\overline{\Omega} \times (0, \tau_0]$. Then $\overline{u}^{\Omega}_{\omega} \leqq Y^{\Omega}_{\overline{\omega}} + \epsilon_0$, in $\overline{\Omega} \times (0, \tau_0]$. From the comparison principle, $\overline{u}^{\Omega}_{\omega} \leqq Y^{\Omega}_{\overline{\omega}} + \epsilon_0$, in $\overline{\Omega} \times (0,T)$. As $\epsilon_0 \to 0$ we get $\overline{u}^{\Omega}_{\omega} \leqq Y^{\Omega}_{\overline{\omega}}$, hence $\overline{u}^{\Omega}_{\omega} = Y^{\Omega}_{\overline{\omega}}$. And any weak solution v of (1.1) with trace $(\overline{\omega}, 0)$ satisfies $v \leqq Y_{k\overline{\omega}}$ in $Q_{\mathbb{R}^N,T}$, for any $k > 1$; thus as $k \to 1$, $\overline{u}_{\omega} \leqq Y_{\overline{\omega}}$, hence $\overline{u}_{\omega} = Y_{\overline{\omega}}$.

(iii) Any weak solution $v \in \mathcal{W}$ is classical since $q \leqq 2$, and from Proposition 3.6, $v(.,t)$ converges uniformly in $\overline{\omega}$ to ∞ as $t \to 0$. Then $\mathcal{W} = \mathcal{C}$. the conclusions follow from (i) and (ii). ■

As a consequence we construct the solution of Theorem 1.3. We are lead to the case $N = 1$.

Proposition 4.6 *Let $q > 1$, $N = 1$. Then there exists a self-similar positive solution $U(x,t) = t^{-a/2} f(t^{-1/2}x)$ of (1.1) in $Q_{\mathbb{R},T}$, with trace $([0,\infty), 0)$, and f satisfies the equation*

$$f''(\eta) + \frac{\eta}{2} f'(\eta) + \frac{a}{2} f(\eta) - |f'(\eta)|^q = 0, \qquad \forall \eta \in \mathbb{R}. \tag{4.9}$$

And setting $c_q = (q')^{-q'}(q-1)^{-1/(q-1)}$,

$$\lim_{\eta \to \infty} \eta^{-q'} f(\eta) = c_q, \tag{4.10}$$

$$\lim_{\eta \to -\infty} e^{\frac{\eta^2}{4}} (-\eta)^{-\frac{3-2q}{q-1}} f(\eta) = C > 0. \tag{4.11}$$

In case $q = 2$, f is given explicitely by

$$f(\eta) = -\ln(\frac{1}{2} erfc(\eta/2)) = -\ln(\frac{1}{2}\int_{\eta/2}^{\infty} e^{-s^2} ds). \tag{4.12}$$

Proof. We apply Theorems 4.1 and 4.5 with $\Omega = \mathbb{R}$ and $\omega = (0,\infty)$. Since ω is starshaped and stable by homothety, we have $Y_{\overline{\omega}}(x,t) = k^a Y_{k\overline{\omega}}(kx, k^2t) = k^a Y_{\overline{\omega}}(kx, k^2t)$ for any $k > 0$. Thus $U = Y_{\overline{\omega}}$ is self-similar. Hence $U(x,t) = t^{-a/2} f(t^{-1/2}x)$, where $\eta \longmapsto f(\eta)$ is a nonnegative C^2-function on $\mathbb{R}$ and satisfies equation (4.9).

In the case $q = 2$, we can compute completely U : The function $V = e^{-U}$ is solution of the heat equation, with $V(0,x) = \chi_{(-\infty,0)}$, thus

$$V(t,x) = (4\pi t)^{-1/2} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} dy = \frac{1}{2}\mathrm{erfc}(\frac{x}{2\sqrt{t}})$$

where $x \longmapsto \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-s^2} ds$ is the complementary error function. Then $U(x,t) = -\ln V$, and f is given by (4.12). Note that f can also be obtained by solving equation $f''(\eta) + \frac{\eta}{2} f'(\eta) - f'(\eta)^2 = 0$, of the first order in f'. We get $f(0) = \ln 2$. As $\eta \to \infty$, since $\mathrm{erfc}(x) = (1/\sqrt{\pi}x)e^{-x^2}(1 + o(1)$, we check that $f(\eta) = (1/4)\eta^2(1 + o(1))$.

Next suppose $q \neq 2$. Writing (4.9) as a system

$$f'(\eta = g(\eta), \qquad g'(\eta) = -\frac{\eta}{2}g(\eta) - \frac{a}{2}f(\eta) + |g(\eta)|^q\,,$$

we obtain that f is positive, from the Cauchy-Lipschitz Theorem. Indeed if there holds $f(\eta_1) = 0$ for some $\eta_1 \in \mathbb{R}$, then $g(\eta_1) = f'(\eta_1) = 0$, thus $(f,g) \equiv (0,0)$. From (3.10), we get $U(1,t) = t^{-a/2}f(t^{-1/2}) \geqq Ct^{1/(q-1)}$, for t small enough, hence $f(\eta) \geqq C\eta^{q'}$ for large η. From (2.14), there holds $U(-1,t) \leqq C_{1,1}e^{-C_{2,1}/t}$ on $(0,\tau_1]$, since U is a pointwise limit of classical solutions with initial data $C_b(\mathbb{R})$ with support in $[0,\infty)$. Then $f(\eta)$ converges to 0 exponentially as $\eta \to -\infty$. Next we show that $f' > 0$ on $\mathbb{R}$: if $f'(\eta_0) = 0$ for some η_0 we have $f''(\eta_0) + \frac{a}{2}f(\eta_0) = 0$. Since $a \neq 0$, η_0 is unique, it is a strict local extremum, which contradicts the behaviour at ∞ and $-\infty$. The universal estimate (2.8) is equivalent to

$$f'^q(\eta) \leqq \frac{1}{q-1}f(\eta), \qquad \forall \eta \in \mathbb{R}. \tag{4.13}$$

Therefore the function $\eta \longmapsto f^{1/q'}(\eta) - c_q^{1/q'}\eta$ is nonincreasing, hence

$$f^{1/q'}(\eta) \leqq c_q^{1/q'}\eta + f^{1/q'}(0), \qquad \forall \eta \geqq 0. \tag{4.14}$$

Otherwise, f is convex: indeed

$$f''' + \frac{\eta}{2}f'' + \frac{1}{2(q-1)}f' - qf'^{q-1}f'' = 0. \tag{4.15}$$

If $f''(\eta_1) = 0$ for some η_1, then $f'''(\eta_1) < 0$, thus η_1 is unique, and $f''(\eta) < 0$ for $\eta > \eta_1$, then f is concave near ∞, which contradicts the estimates above; thus $f''(\eta) > 0$ on $\mathbb{R}$. From (4.9) and (4.13), we deduce that $\eta f' \leqq q'f$.

Let $H(\eta) = \eta^{-q'}f(\eta)$, for $\eta > 0$; then H is nonincreasing, and $H(\eta) \geqq C$ for large η. Thus H has a limit $\lambda > 0$ as $\eta \to \infty$, and $\lambda \leqq c_q$ from (4.14). Let us show that $\lambda = c_q$. Suppose that $\lambda < c_q$. We set $\varphi(\eta) = \eta^{-1/(q-1)}f'(\eta)$, for $\eta > 0$, then $\varphi \leqq q'H$; hence we can find $b < 1$ such that $q\varphi^{q-1}(\eta) < b$ for large η. By computation we find

$$\frac{1}{\eta}\varphi' = \varphi^q - \varphi(\frac{1}{2} + \frac{1}{(q-1)\eta^2}) - \frac{a}{2}H, \tag{4.16}$$

and from (4.15) we obtain

$$\varphi'' + \varphi'(\frac{2}{(q-1)\eta} + \frac{\eta}{2} - q\eta\varphi^{q-1}) + \frac{\varphi}{q-1}(1 - q\varphi^{q-1} + \frac{a}{\eta^2}) = 0$$

If φ is not monotone for large η, then, at any extremal point η,

$$-\varphi'' = \frac{\varphi}{q-1}(1 - q\varphi^{q-1} + \frac{a}{\eta^2}) \geqq \frac{\varphi}{q-1}(1 - b + \frac{a}{\eta^2}),$$

hence $\varphi'' < 0$ for large η, which is impossible. Thus by monotony, φ has a limit θ as $\eta \to \infty$. From the L'Hospital's rule, we deduce that $\lambda = \lim_{\eta\to\infty} f(\eta)/\eta^{q'} = \lim_{\eta\to\infty} f'(\eta)/q'\eta^{1/(q-1)} = \theta/q'$. Then from (4.16), $\lim_{\eta\to\infty}\varphi'(\eta)/\eta = (q'\lambda)^q - \lambda/(q-1)$. Since φ' is integrable, we deduce that $\lambda = c_q$, thus we reach a contradiction. Then (4.10) follows.

Next we study the behaviour near $-\infty$. From (4.13), f and f' converge exponentially to 0. Let $h(\eta) = f'(\eta)/f(\eta)$ for any $\eta \in \mathbb{R}$. Then we find

$$h' + h^2 + \frac{\eta}{2}h + \frac{a}{2} - f'^{(q-1)}h = 0, \tag{4.17}$$

$$h'' + 2hh' + \frac{\eta}{2}h' + \frac{h}{2} - f'^{(q-1)}(qh' + (q-1)h^2) = 0.$$

Either h is not monotone near $-\infty$. At any point where $h' = 0$, we find by computation

$$h'' = (q-1)h(h(h + \frac{\eta}{2}) - \frac{1}{2});$$

hence at any minimal point, $h > |\eta|/2$, then $\lim_{\eta \to -\infty} h(\eta) = \infty$. Let us show that it also true if h is monotone. Suppose that h has a finite limit ℓ, then $\ell = 0$ from (4.17). If $q > 2$, then $\liminf_{\eta \to -\infty} h'(\eta) \geqq |a|/2$, which is contradictory. If $q < 2$, following the method of [16] we write $(e^{\eta^2/4}h)' = e^{\eta^2/4}(-a/2 + o(1))$, then by integration we obtain that $\lim_{\eta \to -\infty} \eta h(\eta) = a$, from the l'Hospital' rule, then $\liminf_{\eta \to \infty} (-\eta)^a f(\eta) > 0$, which is a contradiction. Thus again $\lim_{\eta \to -\infty} h(\eta) = \infty$. And then (4.11) follows as in [16], more precisely, as $\eta \to -\infty$,

$$f(\eta) = Ce^{\frac{-\eta^2}{4}} |\eta|^{\frac{3-2q}{q-1}} (1 - (a-1)(a-2)|\eta|^{-2} + o(|\eta|^{-2})).$$

$\blacksquare$

Remark 4.7 *We have constructed a nonradial solution $f \in C^2(\mathbb{R})$ of (4.9), satisfying (4.10) as $\eta \to \infty$. Let us show that there exists no **radial** solution $f \in C^2(\mathbb{R})$ satisfying the same conditions. Indeed suppose that such a radial solution f_0 exists; then it is still positive, $f_0'(0) = 0$, and 0 is the unique local extremal point of f_0, with $f_0''(0) + \frac{a}{2} f_0(0) = 0$. This is impossible if $q \leqq 2$. Next assume $q > 2$. From the Cauchy-Lipschitz Theorem, setting $C_0 = f_0(0) > 0$ there exists a local unique solution of (4.9) such that $f(0) = C_0$ and $f'(0) = 0$. But the Cauchy problem with initial data $C_0 |x|^{|a|}$ has a self-similar solution of the form $U_{C_0}(x,t) = t^{-a/2} f_{C_0}(t^{-1/2}x)$ with $f_{C_0} \in C^2(\mathbb{R})$ and $f_{C_0}(0) = C_0$, since $|x|^{|a|} \in L^1_{loc}(\mathbb{R})$, see [12, Theorems 1.4,1.5] or Theorem 1.4 below. From local uniqueness, $f_0 = f_{C_0}$, which contradicts its behaviour as $|\eta| \to \infty$.*

Thanks to the barrier function $U(x,t) = t^{-a/2} f(t^{-1/2}x)$ constructed at Proposition 4.6, we obtain more information on the behaviour of the solutions with trace $(\overline{\omega}, 0)$ on the boundary of ω :

Proposition 4.8 *Let $1 < q$, and ω be a smooth open set in $\mathbb{R}^N$. Then the function $Y_{\overline{\omega}}$ constructed at Theorem 4.1 satisfies*

(i) For any $x_0 \in \partial\omega$, $\liminf_{t \to 0} t^{a/2} Y_{\overline{\omega}}(x_0, t) \geqq f(0)$.

(ii) If ω is convex, then for any $x_0 \in \partial\omega$, $\lim_{t \to 0} t^{a/2} Y_{\overline{\omega}}(x_0, t) = f(0)$,

(iii) if $\mathbb{R}^N \backslash \omega$ is convex, then for any $x_0 \in \overline{\omega}$, $\inf_{t > 0} t^{a/2} Y_{\overline{\omega}}(x_0, t) \geqq f(0)$,

where f is defined at Proposition 4.6.

Proof. (i) Since ω is smooth, it satisfies the condition of the interior sphere. Thus we can assume that $x_0 = 0$ and ω contains a ball $B = B(y, \rho)$ with $y = (\rho, 0) \in \mathbb{R}^{N+} = \mathbb{R}^+ \times \mathbb{R}^{N-1}$. Then $Y_{\overline{\omega}} \geqq Y_{\overline{B}}$. Let us consider $Y_{\overline{nB}}(x,t) = n^{-a} Y_{\overline{B}}(x/n, t/n^2)$. The sequence $(Y_{\overline{nB}})$ is nondecreasing, and there holds $Y_{\overline{nB}}(x,t) = 0$ in $B((-1,0),1)$. Thus from estimate (2.9),

$$Y_{\overline{nB}}(x,t) \leqq C(N,q)(t^{-\frac{1}{q-1}}(|x + (1,0)|^{q'} + 1) + t),$$

hence the sequence is locally bounded in $Q_{\mathbb{R}^N,\infty}$. From Theorem 2.7, $(Y_{\overline{nB}})$ converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,\infty})$ to a classical solution u of (1.1). Then u is a solution with trace $(\overline{\mathbb{R}^{N+}}, 0)$, satisfying (4.1), thus $u(x,t) \geqq Y_{\overline{\mathbb{R}^{N+}}}(x,t)$. Observe that $Y_{\overline{\mathbb{R}^{N+}}}(x,t) = U(x_1,t)$, since $U(x_1,t) = \sup_{\varphi \in C_b^+(\mathbb{R}), \mathrm{supp}\varphi \subset \overline{0,\infty}} y_\varphi$, and $Y_{\overline{\mathbb{R}^{N+}}}(x,t) = \sup_{\varphi \in C_c^+(\mathbb{R}^N), \mathrm{supp}\varphi \subset \overline{\mathbb{R}^{N+}}} y_\varphi$. Then $u(0,t) \geqq U(0,t) = f(0)t^{-a/2}$. And $Y_{\overline{nB}}(0,1) = n^{-a} Y_{\overline{B}}(0, 1/n^2)$ converges to $u(0,1) \geqq f(0)$, then $\lim n^{-a} Y_{\overline{B}}(0, 1/n^2) \geqq f(0)$; similarly by replacing $1/n$ by any sequence (ϵ_n) decreasing to 0, then $\liminf_{t \to 0} t^{a/2} Y_{\overline{\omega}}(0,t) \geqq f(0)$.

(ii) Let us show that for any $x_0 \in \partial\omega$, $Y_{\overline{\omega}}(x_0,t) \leqq f(0)t^{-a/2}$. We can assume $x_0 = 0$ and $\omega \subset \mathbb{R}^{N+}$. Then $Y_{\overline{\omega}}(x,t) \leqq Y_{\overline{\mathbb{R}^{N+}}}(x,t) = U(x_1,t)$, hence $Y_{\overline{\omega}}(0,t) \leqq f(0)t^{-a/2}$.

(iii) Since $\mathbb{R}^N \backslash \omega$ is convex, $\overline{\omega}$ is the union of all the tangent half-hyperplanes that it contains. For any such half-hyperplane, we can assume that it is tangent at 0 and equal to $\mathbb{R}^{N+}$. Then for any $x \in \mathbb{R}^{N+}$, there holds $Y_{\overline{\omega}}(x,t) \geqq U(x_1,t) \geqq f(0)$, since f is nondecreasing, and the conclusion follows. $\blacksquare$

5 Existence of solutions with trace $(\mathcal{S}, u_0)$

5.1 Solutions with trace $(\overline{\omega} \cap \Omega, u_0)$, ω open

Proof of Theorem 1.4. (i) Approximation and convergence. We define suitable approximations of the initial trace $(\mathcal{S}, u_0)$ according to the value of q. We consider a sequence (φ_p) in $C_b\left(\mathbb{R}^N\right)$ (resp. $C_0\left(\overline{\Omega}\right)$) as in the proof of Theorem 4.1. We define a sequence (ψ_p) in the following way: if $1 < q < q_*$, we define ψ_p by the restriction of the measure u_0 to $\mathcal{R}_{1/p}^{int} \cap B_p$ (resp. to $\mathcal{R}_{1/p}^{int} \cap \Omega_{1/p}^{int}$); if $q_* \leqq q \leqq 2$, we take $\psi_p = \inf(u_0, p)\chi_{\mathcal{R}\cap B_p}$ (resp. $\psi_p = \inf(u_0, p)\chi_{\mathcal{R}}$). If $q > 2$, by our assumption we can take a nondecreasing sequence (ψ_p) in $C_c(\mathcal{R})$ converging to u_0 in $L^1_{loc}(\mathcal{R})$. We set $u_{0,p} = \varphi_p + \psi_p$. Then for $1 < q < q_*$, $u_{0,p} \in \mathcal{M}_b^+(\Omega)$, for $q_* \leqq q \leqq 2$, $u_{0,p} \in L^r(\Omega)$ for any $r > 1$ and for $q > 2$, $u_{0,p} \in C_b\left(\mathbb{R}^N\right)$. In any case there exists a solution u_p of (1.1) (resp. of $(D_{\Omega,T})$) with initial data $u_{0,p}$, unique among the weak solutions if $q \leqq 2$, see Theorem 2.12, and among the classical solutions in $C\left([0,T) \times \overline{\Omega}\right)$ if $q > 2$, and the sequence (u_p) is nondecreasing if $q \geqq q_*$.

Moreover if $\Omega = \mathbb{R}^N$, (u_p) satisfies the estimate (2.9): considering a ball $B(x_0, \eta) \subset \mathbb{R}^N \backslash \overline{\omega}$, there exists $C = C(N, q, \eta)$ such that for $p \geqq p(\eta)$ large enough,

$$u_p(x,t) \leqq C(t^{-\frac{1}{q-1}}(|x-x_0|^{q'}+1) + t + \int_{B(x_0,\eta)} du_{0,p}) \leqq C(t^{-\frac{1}{q-1}}(|x-x_0|^{q'}+1) + t + \int_{B(x_0,\eta)} du_0),$$

then (u_p) is uniformly locally bounded in $Q_{\mathbb{R}^N,T}$ (resp. if Ω is bounded, (u_p) satisfies (2.5), since it is constructed by approximation from solutions with smooth initial data). From Theorem 2.7 (resp. 2.6)), we can extract a subsequence $C^{2,1}_{loc}$-converging to a classical solution u of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$). If $q \geqq q_*$, from uniqueness, (u_p) is nondecreasing, then (u_p) converges to $u = \sup u_p$.

(ii) Behaviour of u in $\overline{\omega}$. By construction, $u \geqq Y_{\overline{\omega}}$, (resp. $u \geqq Y_{\overline{\omega}}^{\Omega}$), then u satisfies (4.2), hence as $t \to 0$, $u(.,t)$ converges uniformly to ∞ on any compact in ω, thus (1.9) holds; if $q < q_*$, u satisfies (4.4), thus the convergence is uniformly on $\overline{\omega} \cap \Omega$.

(iii) Behaviour of u in $\mathcal{R}$. From (3.5) and (3.4), for any $\xi \in C^{1,+}(\mathbb{R}^N)$, with support in $\mathcal{R}$,

$$\int_{\mathbb{R}^N} u_p(.,t)\xi^{q'}dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^N} |\nabla u_p|^q \xi^{q'}dx \leqq Ct\int_{\mathbb{R}^N} |\nabla \xi|^{q'}dx + \int_{\mathbb{R}^N} \xi^{q'}d\psi_p, \tag{5.1}$$

$$\int_\Omega u_p(.,t)\xi dx + \int_0^t \int_\Omega (\nabla u_p.\nabla\xi + |\nabla u_p|^q\, \xi)dxdt = \int_\Omega \xi du_{0,p}. \tag{5.2}$$

First suppose $q < q_*$. From Theorem 2.11, $(|\nabla u_p|^q)$ is equi-integrable in $Q_{K,\tau}$ for any compact set $K \subset \mathcal{R}$ and $\tau \in (0,T)$. From (5.2) for any $\zeta \in C_c(\mathcal{R})$, for $p = p(\zeta)$ large enough such that the support of ζ is contained in $\mathcal{R}_{1/p}^{int} \cap B_p$ (resp. $\mathcal{R}_{1/p}^{int} \cap \Omega_{1/p}^{int}$),

$$\int_{\mathcal{R}} u_p(t,.)\zeta dx + \int_0^t \int_{\mathcal{R}} |\nabla u_p|^q \zeta dx = -\int_0^t \int_{\mathcal{R}} \nabla u_p.\nabla\zeta dx + \int_{\mathcal{R}} \zeta du_0.$$

Then we can go to the limit as $p \to \infty$:

$$\int_{\mathcal{R}} u(t,.)\zeta dx + \int_0^t \int_{\mathcal{R}} |\nabla u|^q \zeta dx = -\int_0^t \int_{\mathcal{R}} \nabla u.\nabla\zeta dx + \int_{\mathcal{R}} \zeta du_0.$$

thus $\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\zeta dx = \int_{\mathbb{R}^N} \zeta du_0$.

Next suppose $q_* \leqq q \leqq 2$ and $u_0 \in L^1_{loc}(\mathcal{R})$, or $q > 2$ and u_0 is limit of a sequence of nondecreasing continuous functions. Then $\psi_p \leqq u_0$. From (5.1), we have $|\nabla u|^q \in L^1_{loc}\left([0,T); L^1_{loc}(\mathcal{R})\right)$ from the Fatou Lemma. Hence, from Lemma 3.1, u admits a trace $\mu_0 \in \mathcal{M}(\mathcal{R})$. For any fixed $\zeta \in C_c^+(\mathcal{R})$, we $\lim_{t\to 0} \int_{\mathbb{R}^N} u_p(.,t)\zeta dx = \int_{\mathcal{R}} \zeta\psi_p dx$. Since (u_p) is *nondecreasing*, we get

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\zeta dx = \int_{\mathcal{R}} \zeta d\mu_0 \geqq \lim_{t\to 0} \int_{\mathbb{R}^N} u_p(.,t)\zeta dx = \int_{\mathcal{R}} \zeta\psi_p dx.$$

thus from the Beppo-Levy Theorem, $\mu_0 \geqq u_0$. Moreover for any $\zeta \in C_c(\mathcal{R})$, from (5.2),

$$\int_{\mathcal{R}} u_p(t,.)\zeta dx + \int_0^t \int_{\mathcal{R}} |\nabla u_p|^q \zeta dx = \int_0^t \int_{\mathcal{R}} u_p \Delta\zeta dx + \int_{\mathcal{R}} \zeta\psi_p dx;$$

and (u_p) is bounded in $L^k(Q_{K,\tau})$ for any $k \in [1, q_*)$, for any compact set $K \subset \mathcal{R}$, and $u_p \to u$ *a.e.* in $\mathcal{R}$, then (u_p) converges strongly in $L^1(Q_{K,\tau})$, thus from the dominated convergence Theorem and the Fatou Lemma,

$$\int_{\mathcal{R}} u(t,.)\zeta dx + \int_0^t \int_{\mathcal{R}} |\nabla u|^q \zeta dx \leqq \int_0^t \int_{\mathcal{R}} u \Delta\zeta dx + \int_{\mathcal{R}} \zeta du_0.$$

But from Lemma 3.1,

$$\int_{\mathcal{R}} u(t,.)\zeta dx + \int_0^t \int_{\mathcal{R}} |\nabla u|^q \zeta dx = \int_0^t \int_{\mathcal{R}} u \Delta\zeta dx + \int_{\mathcal{R}} \zeta d\mu_0,$$

then $\int_{\mathcal{R}} \zeta d\mu_0 \leqq \int_{\mathcal{R}} \zeta du_0$, hence $\mu_0 \leqq u_0$, hence $\mu_0 = u_0$.

In any case u admits the trace $(\mathcal{S}, u_0)$. ■

5.2 Solutions with any Borel measure

In this part we consider the subcritical case with an arbitrary closed set $\mathcal{S}$.

Theorem 5.1 *Let $1 < q < q_*$, and $\Omega = \mathbb{R}^N$ (resp. Ω bounded). Let $\mathcal{S}$ be a closed set in Ω, such that $\mathcal{R} = \Omega \backslash \mathcal{S}$ is nonempty. Let $u_0 \in \mathcal{M}^+(\mathcal{R})$.*

(i) Then there exists a solution u of (1.1) (resp. of $(D_{\Omega,T})$) with initial trace $(\mathcal{S}, u_0)$, such that u satisfies (4.4), hence $u(t,.)$ converges to ∞ uniformly on $\mathcal{S}$.

(ii) There exists a minimal solution $u_{\min}$, satisfying the same conditions.

Proof. Assume that $\Omega = \mathbb{R}^N$ (resp. Ω bounded) (i) Existence of a solution. Let $B(x_0,\eta) \subset \Omega\backslash\mathcal{S}$, and δ_0 small enough such that $B(x_0,\eta) \subset \Omega\backslash\mathcal{S}_{\delta_0}^{ext}$. For any $\delta \in (0,\delta_0)$ we can suppose that $\mathcal{S}_\delta^{ext} = \overline{\omega_\delta} \cap \Omega$, where ω_δ is a smooth open subset of Ω (if $\mathcal{S}_\delta^{ext}$ is not smoothenough regular, we replace it by a smooth open set $\mathcal{S}_\delta'^{ext}$such that $\mathcal{S} \subset \mathcal{S}_\delta'^{ext} \subset \mathcal{S}_\delta^{ext}$). Let u_δ be the solution with initial trace $(\mathcal{S}_\delta^{ext}, u_0 \llcorner (\Omega\backslash\mathcal{S}_\delta^{ext}))$ constructed at Theorem 1.4. Then u_δ admits the trace u_0 on $B(x_0,\eta)$, thus it also satisfies the estimates (2.9) (resp. (2.5)), thus $(u_\delta)_{\delta<\delta_0}$ is uniformly locally bounded in $Q_{\Omega,T}$. From Theorem 2.7 (resp. 2.6), one can extract a subsequence converging in $C_{loc}^{2,1}(Q_{\Omega,T})$ to a solution u of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$). As in the proof of Theorem 1.4, for any compact $K \subset \mathcal{R}$, taking $\delta < \delta_K$ small enough so that $K \subset \Omega\backslash\mathcal{S}_{\delta_K}^{ext}$, and choosing a test function ξ with compact support in K in $\mathcal{R}$, we obtain that $(|\nabla u_\delta|^q)_{\delta<\delta_K}$ is equi-integrable in $Q_{K,\tau}$ for any $\tau \in (0,T)$. Then we get for any $\xi \in C_c(\mathcal{R})$,

$$\int_{\mathbb{R}^N} u(t,.)\xi dx + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \xi dx = -\int_0^t \int_{\mathbb{R}^N} \nabla u.\nabla\xi dx + \int_{\mathbb{R}^N} \xi du_0.$$

thus $\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\xi dx = \int_{\mathbb{R}^N} \xi du_0$. Moreover for any $x_0 \in \mathcal{S}$, $u_\delta \geqq Y_{\{x_0\}}$ in $Q_{\mathbb{R}^N,T}$, (resp. $u_\delta \geqq Y_{\{x_0\}}^\Omega$ in $Q_{\Omega,T}$) from Proposition 3.6, hence the same happens for u, which implies (1.9). Thus u admits $(\mathcal{S}, u_0)$ as initial trace, and $u(.,t)$ converges uniformly on $\mathcal{S}$ to ∞ as $t \to 0$.

(ii) Existence of a **minimal** solution.

Assume that $\Omega = \mathbb{R}^N$. Let A be the set of solutions with initial trace $(\mathcal{S}, u_0)$. We consider for fixed $\epsilon > 0$, the Dirichlet problem in $Q_{B_p,T}$, $p \geqq 1$, with initial data $m(x,\epsilon) = \inf_{v\in A} v(x,\epsilon)$. Thus $0 \leqq m(x,\epsilon) \leqq u(x,\epsilon)$, where u has been defined at step (i), and $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$, thus $m(.,\epsilon) \in L_{loc}^1(\mathbb{R}^N)$. Since $m \in L^1(B_p)$, there exists a unique solution $w_{p,\epsilon}$ of $(D_{B_p,T})$ with initial data $m(x,\epsilon)$ in B_p. From Corollary 2.15, $w_{p,\epsilon}(x,t) \leqq$

$v(x,t+\epsilon)$ for any $v\in A$ and $x\in B_p$. Moreover for any $v\in A$ and any $x_0\in\mathcal{S}$, there holds $v\geqq Y_{\{x_0\}}\geqq Y_{x_0}^{B_p}$, thus $m(x,\epsilon)\geqq Y_{x_0}^{B_p}(x,\epsilon)$, hence $w_{p,\epsilon}(x,t)\geqq Y_{x_0}^{B_p}(x,t+\epsilon)$, from [27, Proposition 2.1]. For any $z_0\in B_p$ and $\gamma>0$ such that $\mathcal{B}=B(z_0,\gamma)$ satisfies $\overline{\mathcal{B}}\subset\mathcal{R}\cap B_p$, let w_U be the unique solution of the Dirichlet problem in $\mathcal{B}$ with initial data $u_0\llcorner\mathcal{B}$. Then from Corollary 2.15, $v(x,t)\geqq w_{\mathcal{B}}(x,t)$ in $Q_{\mathcal{B},T}$, for any $v\in A$, thus $m(x,\epsilon)\geqq w_{\mathcal{B}}(x,\epsilon)$, thus $w_{p,\epsilon}(x,t)\geqq w_{\mathcal{B}}(x,t+\epsilon)$.

Next we go to the limit as $\epsilon\to 0$. From Theorem 2.6, one can extract a subsequence, still denoted $(w_{p,\epsilon})$, converging *a.e.* to a solution w_p of the Dirichlet problem $(D_{B_p,T})$. And in B_p (with the notations above), $w_p\leqq v$ for any $v\in A, w_p\geqq Y_{x_0}^{B_p}$ and $w_p\geqq w_U$. Finally we go to the limit as $p\to\infty$. Since u is locally bounded, then (w_p) is uniformly locally bounded. From Theorem 2.7, one can extract a subsequence converging in $C_{loc}^{2,1}(Q_{\mathbb{R}^N,T})$ to a weak solution denoted $u_{\min}$ of (1.1) in $Q_{\mathbb{R}^N,T}$. Then $u_{\min}$ satisfies $u_{\min}\leqq v$ for any $v\in A$, and $u_{\min}\geqq Y_{x_0}^{B_p}$ for any $x_0\in\mathcal{S}$, and $u_{\min}\geqq w_U$ for any $z_0\in\mathcal{R}$ and $\gamma>0$ such that $\mathcal{B}=B(z_0,\gamma)$ satisfies $\overline{\mathcal{B}}\subset\mathcal{R}$. As a consequence $u_{\min}$ satisfies the trace condition (1.9) on $\mathcal{S}$. And for any $z_0\in\mathcal{R}$, and any $\xi\in C_c^0(\mathcal{R})$ with support in U,

$$\int_{\mathcal{R}}u(.,t)\xi dx\geqq\int_{\mathcal{R}}u_{\min}(.,t)\xi dx\geqq\int_{\mathcal{R}}w_U(.,t)\xi dx$$

hence

$$\lim_{t\to 0}\int_{\mathcal{R}}u_{\min}(.,t)\xi dx=\int_{\mathcal{R}}\xi du_0.$$

Then $u_{\min}$ admits the trace $(\mathcal{S},u_0)$. Thus $u_{\min}$ is minimal, and $u_{\min}=\min_{v\in A}v$.

Assume that Ω is bounded. The proof still works with B_p replaced by Ω, which requires only to go to the limit in ε and use Theorem 2.6. ∎

In the case where u_0 is a *bounded* measure we can give more convergence results:

Corollary 5.2 *Under the assumptions of Theorem 5.1 suppose that $u_0\in\mathcal{M}_b^+(\mathcal{R})$. Then for any $\varphi\in C_b(\Omega)$ with support in $\mathcal{R}$, $u(.,t)\varphi\in L^1(\mathcal{R})$ for any $t\in(0,T)$, and*

$$\lim_{t\to 0}\int_{\mathcal{R}}u(.,t)\varphi dx=\int_{\mathcal{R}}\varphi du_0,\tag{5.3}$$

and similarly for $u_{\min}$. More precisely, if $\Omega=\mathbb{R}^N$, (5.3) is valid for any weak solution v of (1.1) with trace $(\mathcal{S},u_0)$.

Proof. First assume that $\Omega=\mathbb{R}^N$ and v is any weak solution with trace $(\mathcal{S},u_0)$ let $\psi\in C_b^1(\mathbb{R}^N)$ with support in $\mathcal{R}$, and $\varphi_n\in\mathcal{D}\left(\mathbb{R}^N\right)$ with values in $[0,1]$, with $\varphi_n=1$ on B_n, 0 on B_{2n}, and $(|\nabla\varphi_n|)$ bounded. Then from (3.5),

$$\int_{\mathcal{R}}v(.,t)(\psi\varphi_n)^{q'}dx\leqq C(q)t\int_{\mathbb{R}^N}|\nabla(\psi\varphi_n)|^{q'}dx+\int_{\mathbb{R}^N}(\psi\varphi_n)^{q'}du_0\leqq Ct+\int_{\mathcal{R}}\psi^{q'}du_0;$$

thus $v(.,t)\psi^{q'}\in L^1(\mathcal{R})$, and $\limsup_{t\to 0}\int_{\mathcal{R}}v(.,t)\psi^{q'}dx\leqq\int_{\Omega}\psi^{q'}du_0$ from the Fatou Lemma. And

$$\liminf_{t\to 0}\int_{\mathcal{R}}v(.,t)\psi^{q'}dx\geqq\lim_{t\to 0}\int_{\mathcal{R}}v(.,t)(\psi\varphi_n)^{q'}dx=\int_{\mathcal{R}}(\psi\varphi_n)^{q'}du_0,$$

thus from the Beppo-Levy Theorem, we get (5.3) by density.

Next suppose that Ω is bounded, note that u can be obtained as a limit in $C_{loc}^{2,1}(Q_{\Omega,T})\cap C_{loc}^{1,0}\left(\overline{\Omega}\times(0,T)\right)$ of classical solutions u_n with smooth data $u_{n,0}=u_{n,0}^1+u_{n,0}^2$ with supp$u_{n,0}^1\subset\overset{\circ}{\mathcal{S}}{}_{3\delta_0}^{ext}$, supp$u_{n,0}^1\subset\mathcal{R}$, and $(u_{n,0}^1)$ converges to u_0 weakly in $\mathcal{M}_b(\mathcal{R})$. For any nonnegative $\xi\in C_b^1(\Omega)$ with support in $\mathcal{R}$,

$$\int_{\mathcal{R}}u_n(.,t)\xi^{q'}dx\leqq C(q)t\int_{\mathcal{R}}|\nabla\xi|^{q'}dx+\int_{\Omega}\xi^{q'}u_{n,0}^2dx,$$

from Remark 3.4, hence

$$\int_{\mathcal{R}} u(.,t)\xi^{q'}dx \leqq C(q)t\int_{\mathcal{R}} |\nabla\xi|^{q'}dx + \int_{\Omega} \xi^{q'}du_0,$$

and then $\limsup_{t\to 0}\int_{\mathcal{R}} u(.,t)\psi^{q'}dx \leqq \int_{\Omega}\psi^{q'}du_0$. And for any $\varphi_n \in \mathcal{D}(\Omega)$ with values in $[0,1]$, with $\varphi_n = 1$ on $\mathcal{R}^{int}_{1/n}$,

$$\liminf_{t\to 0}\int_{\mathcal{R}} u(.,t)\psi^{q'}dx \geqq \lim_{t\to 0}\int_{\mathcal{R}} v(.,t)(\psi\varphi_n)^{q'}dx = \int_{\mathcal{R}} (\psi\varphi_n)^{q'}du_0,$$

Thus u still satisfies (5.3). The same happens for $u_{\min}$, since $\limsup_{t\to 0}\int_{\mathcal{R}} u_{\min}(.,t)\varphi dx \leqq \int_{\mathcal{R}}\varphi du_0$ and $\liminf_{t\to 0}\int_{\mathcal{R}} u_{\min}(.,t)\psi^{q'}dx \geqq \int_{\mathcal{R}}(\psi\varphi_n)^{q'}du_0$. ■

Remark 5.3 *Assume $1 < q < q_*$. Note some consequences of Theorems 5.1 and 1.4.*

(i) For any constant $C > 0$, there exists a minimal solution u_C with trace $(\{0\}, C|x|^{-a}))$. Then u_C is radial and self-similar. This shows again the existence of the solutions of example 2, Section 3. This shows that the set $\{C(\beta) : \beta > F(0)\}$, where F and $C(\beta)$ ere defiend at (1.5) and (3.9), is equal to $(0,\infty)$.

(ii) Suppose $N = 1$. For any $C > 0$ there exists a minimal solution $\widetilde{u}_C$ with trace $([0,\infty), C(x^-)^{-a})$; it is self-similar, $\widetilde{u}_C(x,t) = t^{-a/2}\widetilde{f}(t^{-1/2}x)$; as in the proof of Proposition 4.6, we obtain that $\widetilde{f}$ is increasing and $\lim_{\eta\to\infty}\widetilde{f}(\eta)\eta^{-q'} = c$, and $\lim_{\eta\to-\infty}\eta\widetilde{f}'(\eta)/\widetilde{f}(\eta) = a$, and then $\lim_{\eta\to-\infty}\widetilde{f}(\eta)|\eta|^a = C$. In the same way, for any $C > 0$, there exists a minimal solution $\widehat{u}_C$ with trace $(\{0\}, C(x^+)^{-a})$; then it is self-similar, $\widehat{u}_C(x,t) = t^{-a/2}\widehat{f}(t^{-1/2}x)$, where $\eta \longmapsto \widehat{f}(\eta)$ is defined on $\mathbb{R}$, and we check that $\widehat{f}$ has an exponential decay at $-\infty$, and $\lim_{\eta\to\infty}\widehat{f}(\eta)\eta^a = C$.

Next we look for a maximal solution when the measure u_0 is bounded. A crucial point in case $\Omega = \mathbb{R}^N$ is the obtention of an upper estimate, based on Theorems 2.8 and 2.9:

Proposition 5.4 *$1 < q \leqq 2$. Let $\mathcal{S}$ be a compact set in $\mathbb{R}^N$, and $u_0 \in \mathcal{M}^+(\mathbb{R}^N\backslash\mathcal{S})$, bounded at ∞. Then any weak solution v of (1.1) in $Q_{\mathbb{R}^N,T}$ with trace $(\mathcal{S}, u_0)$ satisfies, for any $0 < \epsilon < \tau < T$,*

$$\|v\|_{L^\infty((\epsilon,\tau);L^\infty(\mathbb{R}^N))} \leqq C, \qquad C = C(N,q,\epsilon,\tau). \tag{5.4}$$

Proof. Let $\tau \in 0,T)$. We take $\eta = 1$ and $x_0 \in \mathbb{R}^N\backslash\mathcal{S}_1$ in (2.9). Then for any $(x,t) \in Q_{\mathbb{R}^N,\tau}$,

$$v(x,t) \leqq C(q)t^{-\frac{1}{q-1}}|x-x_0|^{q'} + C(N,q)(t^{-\frac{1}{q-1}} + t + \int_{B(x_0,1)} du_0). \tag{5.5}$$

In particular it holds in $\mathcal{S}_2 \times (0,\tau]$. And for any $(x,t) \in \mathbb{R}^N\backslash\mathcal{S}_2$, since $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N\backslash\mathcal{S}_1)$, from (2.7),

$$v(x,t) \leqq C(N,q,\tau)t^{-N/2}(t + \int_{B(x_0,1)} du_0) \leqq C(N,q,\tau)t^{-N/2}(t + \int_{\mathbb{R}^N\backslash\mathcal{S}_1} du_0). \tag{5.6}$$

Then (5.4) follows. ■

Theorem 5.5 *Let $1 < q < q_*$. Let $\Omega = \mathbb{R}^N$ (resp. Ω bounded). Assume that $\mathcal{S}$ is compact in Ω and $u_0 \in \mathcal{M}_b^+(\Omega)$ with support in $\mathcal{R}\cup\overline{\Omega}$, where $\mathcal{R} = \Omega\backslash\mathcal{S}$. Then there exists a maximal solution u of (1.1) (resp. of $(D_{\Omega,T})$) among the solutions with trace $(\mathcal{S}, u_0)$ (resp. among the solutions v of trace $(\mathcal{S}, u_0)$ such that $v(.,t)$ converges weakly in $\mathcal{R}$ to u_0 as $t \to 0$).*

Proof. Assume $\Omega = \mathbb{R}^N$ (resp. Ω bounded). Let $\delta > 0$ be fixed, such that $\delta < d(\mathcal{S},\text{supp}u_0)/3$, hence $\text{supp}u_0 \subset \Omega\backslash\mathcal{S}_{3\delta}$. Let u_δ be the solution with initial trace $(\mathcal{S}^{ext}_\delta, u_0)$ constructed at Theorem 1.4.

Let v be any weak solution with trace $(\mathcal{S}, u_0)$ (resp. and such that $v(.,t)$ converges weakly in $\mathcal{M}_b(\mathcal{R})$. Then $v(.,t) \leqq C(N,q,\delta)t$ in $\mathcal{K}_\delta = \overset{\circ}{\mathcal{S}}{}^{ext}_{5\delta/2} \backslash \mathcal{S}^{ext}_{\delta/2}$, from Lemma 2.16 (resp. from (2.15) in $\mathcal{O} = \overset{\circ}{\mathcal{S}}{}^{ext}_{3\delta} \backslash \mathcal{S}^{ext}_{\delta}$, valid since $v \in C([0,T) \times \mathcal{O})$). Let $\epsilon_0 > 0$. Then there exists $\tau_0 = \tau_0(\epsilon_0, \delta) < T$ such that $v(.,t) \leqq \epsilon_0$ in $\mathcal{K}_\delta \times (0, \tau_0]$. Let $\epsilon < \tau_0$, and $C_\epsilon = \max_{\mathcal{S}_{2\delta}} v(.,\epsilon)$. Since u_δ converges to ∞ uniformly on the compact sets of $\mathcal{S}^{ext}_\delta$, there exists $\tau_\epsilon < \tau_0$ such that for any $\theta \in (0, \tau_\epsilon)$, $u_\delta(.,\theta) \geqq C_\epsilon \geqq v(.,\epsilon)$ in $\mathcal{S}_{\delta/2}$. Since $v(.,\epsilon) \leqq \epsilon_0$ in $\mathcal{K}_\delta$, there holds $v(.,\epsilon) \leqq u_\delta(.,\theta) + \epsilon_0$ in $\mathcal{S}_{2\delta}$. And $v(.,t) \leqq \epsilon_0$ on $\partial\mathcal{S}_{2\delta} \times (0, \tau_0]$, thus $v(.,t+\epsilon) \leqq u_\delta(.,t+\theta) + \epsilon_0$ in $\mathcal{S}_{2\delta} \times (0, \tau_0 - \epsilon]$ from the comparison principle. As $\theta \to 0$, then $\epsilon \to 0$, we get

$$v(.,t) \leqq u_\delta(.,t) + \epsilon_0 \quad \text{ in } \mathcal{S}_{2\delta} \times (0, \tau_0]. \tag{5.7}$$

Otherwise, since $u_0 \in \mathcal{M}_b^+(\Omega)$, there exists a unique solution w of $(P_{\Omega,T})$ with initial data u_0, from Theorem 2.12. We claim that

$$v(x,t) \leqq w(x,t) + \epsilon_0, \qquad \text{ in } \overline{\Omega \backslash \mathcal{S}_{2\delta}} \times (0, \tau_0]. \tag{5.8}$$

Indeed let $\varphi_\delta \in C(\overline{\Omega})$ with values in $[0,1]$ with support in $\overline{\Omega} \backslash \mathcal{S}_{2\delta}$ and $\varphi_\delta = 1$ on $\mathbb{R}^N \backslash \mathcal{S}_{5\delta/2}$. From Proposition 5.4 (resp. from Theorem 2.12), the function $x \longmapsto v(x, \tau_0/n)$ is bounded, and continuous. Let $w_{\delta,n}$ be the solution of (1.1) in $Q_{\Omega,T}$ with initial data $v(., \tau_0/n)\varphi_\delta$. As $n \to \infty$, $v(., \tau_0/n)\varphi_\delta$ converges to $u_0\varphi_\delta = u_0$ weakly in $\mathcal{M}_b\left(\mathbb{R}^N\right)$, from Remark 5.2 (resp. from our assumption). Hence $w_{\delta,n}$ converges to w, from Proposition 2.14. And then

$$v(., \tau_0/n) = v(., \tau_0/n)\varphi_\delta + v(., \tau_0/n)(1 - \varphi_\delta) \leqq w_{\delta,n}(.,0) + \epsilon_0$$

in $\overline{\Omega \backslash \mathcal{S}_{2\delta}}$, and on the lateral boundary of $\overline{\Omega \backslash \mathcal{S}_{2\delta}} \times (0, \tau_0(1 - 1/n)]$, there holds $v(x, t + \tau_0/n) \leqq \epsilon_0$. Then $v(x, t + \tau_0/n) \leqq w_{\delta,n}(.,t) + \epsilon_0$ in $\overline{\Omega \backslash \mathcal{S}_{2\delta}} \times (0, \tau_0(1 - 1/n)]$. As $n \to \infty$, we deduce (5.8).

Next we get easily that $w \leqq u_\delta$ on $\overline{\Omega \backslash \mathcal{S}_{2\delta}} \times (0, \tau_0]$, by considering their approximations, hence

$$v(x,t) \leqq u_\delta(x,t) + \epsilon_0, \qquad \text{ in } \overline{\Omega \backslash \mathcal{S}_{2\delta}} \times (0, \tau_0]. \tag{5.9}$$

As a consequence, from (5.7) and 5.9),

$$v(x,t) \leqq u_\delta(x,t) + \epsilon_0, \qquad \text{ in } \overline{\Omega} \times (0, \tau_0].$$

The last step is to prove that the inequality holds up to time T. We can apply the comparison principle because, from Proposition 5.4, u and $v \in C_b((\epsilon, T); C_b(\mathbb{R}^N)$ for any $\epsilon > 0$ (resp. because v and u_δ are classical solutions of $(D_{\Omega,T})$). Then

$$v(x,t) \leqq u_\delta(x,t) + \epsilon_0, \qquad \text{ in } \overline{\Omega} \times (0, T)$$

As $\epsilon_0 \to 0$, we deduce that $v \leqq u_\delta$. Finally as $\delta \to 0$, up to a subsequence, $\{u_\delta\}$ converges to a solution u of (1.1) (resp. of $(D_{\Omega,T})$, such that $v \leqq u$, thus u satisfies (1.9). As in Theorem 1.4, by integrability of $(|\nabla u_\delta|^q)$ we obtain that u admits the trace u_0 in $\mathcal{R}$, thus u has the trace $(\mathcal{S}, u_0)$ (resp. and the convergence holds weakly in $\mathcal{M}_b(\mathcal{R})$). Thus u is maximal. ■

From Theorems 5.1 and 5.5, this ends the proof of Theorem 1.5.

6 The case $0 < q \leqq 1$

Notice that Theorem 2.5 is also valid for $q = 1$. In fact it can be improved when q is subcritical, and extended to the case $q < 1$.

Theorem 6.1 *(i) Let $0 < q < q$, and Ω be any domain in $\mathbb{R}^N$. Let u be any (signed) weak solution of (1.1) in $Q_{\Omega,T}$. Then $u \in C^{2+\gamma, 1+\gamma/2}_{loc}(Q_{\Omega,T})$ for some $\gamma \in (0,1)$. If Ω is bounded, any weak solution u of problem $(D_{\Omega,T})$ satisfies $u \in C^{1,0}\left(\overline{\Omega} \times (0,T)\right) \cap C^{2+\gamma, 1+\gamma/2}_{loc}(Q_{\Omega,T})$ for some $\gamma \in (0,1)$.*

(ii) Let $0 < q \leqq 1$ and Ω bounded. For any sequence of weak nonnegative solutions (u_n) of $(D_{\Omega,T})$, bounded in $L^\infty_{loc}((0,T);L^1(\Omega))$ one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}\left(\overline{\Omega} \times (0,T)\right)$ to a weak solution u of $(D_{\Omega,T})$.

Proof. (i) From our assumptions, $u \in C((0,T);L^1_{loc}(Q_{\Omega,T}))$, thus $u \in L^\infty_{loc}((0,T);L^1_{loc}(Q_{\Omega,T}))$. We can write (1.1) under the form $u_t - \Delta u = f$, with $f = -|\nabla u|^q$. From Theorem 2.11 $u \in L^1_{loc}((0,T);W^{1,k}_{loc}(\Omega)$ for any $k \in [1, q_*)$ and satisfies (2.10).

First suppose $q \leqq 1$. We choose $k \in (1, q_*)$, thus $(|\nabla u| + |u|) \in L^k_{loc}(Q_{\Omega,T})$. Then $u \in \mathcal{W}^{2,1,k}_{loc}(Q_{\Omega,T})$, see [22, theorem IV.9.1]. From the Gagliardo-Nirenberg inequality, for almost any $t \in (0,T)$,

$$\|\nabla u(.,t)\|_{L^{kq_*}(\omega)} \leqq c\|u(t)\|^{\frac{1}{q_*}}_{W^{2,k}(\omega)}\|u(t)\|^{1-\frac{1}{q_*}}_{L^1(\omega)},$$

where $c = c(N,s,\omega)$. Hence we obtain $|\nabla u| \in L^{kq_*}_{loc}(\Omega)$. In the same way

$$\|u(.,t)\|_{L^{kq_*}(\omega)} \leqq c\|u(t)\|^{\theta}_{W^{2,s}(\omega)}\|u(t)\|^{1-\theta}_{L^1(\omega)},$$

with $\theta = (1 - 1/kq_*)/((N+2)/N - 1/s) < 1$. Therefore $|u| \in L^{sq_*}_{loc}(\Omega)$. Then $u \in \mathcal{W}^{2,1,kq_*}_{loc}(Q_{\Omega,T})$. By induction $u \in \mathcal{W}^{2,1,k(q_*)^n}_{loc}(Q_{\Omega,T})$ for any $n \geqq 1$. Choosing n such that $k(q_*)^n > N+2$, we deduce that $|\nabla u| \in C^{\delta,\delta/2}(Q_{\omega,s,\tau})$ for any $\delta \in (0, 1-(N+2)/s(q_*)^n)$, see [22, Lemma II.3.3]. Then $f \in C^{\delta q,\delta q/2}_{loc}(Q_{\Omega,T})$, thus $u \in C^{2+\delta q,1+\delta q/2}(Q_{\omega,s,\tau})$.

Next suppose $1 < q < q_*$. we choose $k \in (1, q_*/q)$, hence $(|\nabla u|^q + |u|) \in L^k_{loc}(\Omega)$; as above, $|\nabla u| + |u| \in L^{kq_*}_{loc}(\Omega)$, hence $(|\nabla u|^q + |u|) \in L^{kq_*/q}_{loc}(\Omega)$; then $u \in \mathcal{W}^{2,1,kq_*/q}_{loc}(Q_{\Omega,T})$. By induction we get again that $|\nabla u| \in C^{\delta,\delta/2}_{loc}(Q_{Q,T})$ for some $\delta \in (0,1)$, then $f \in C^{\gamma,\gamma/2}_{loc}(Q_{Q,T})$ for some $\gamma \in (0,1)$, thus $u \in C^{2+\gamma,1+\gamma/2}_{loc}(Q_{\Omega,T})$ for some $\gamma \in (0,1)$.

If Ω is bounded, and u is a weak solution of $(D_{\Omega,T})$, then u satisfies (2.11). In the same way, $u \in \mathcal{W}^{2,1,k}(Q_{\Omega,s,\tau})$, and by induction $u \in C^{1,0}\left(\overline{\Omega} \times (0,T)\right) \cap C^{2+\gamma,1+\gamma/2}_{loc}(Q_{\Omega,T})$.

(ii) From (2.11), $\|u\|_{C^{1,0}(\overline{Q_{\Omega,s,\tau}})} + \|\nabla u\|_{C^{\gamma,\gamma/2}(\overline{Q_{\Omega,s,\tau}})}$ is bounded in terms of $\||\nabla u|^q\|_{L^1(Q_{\omega,s,\tau})} + \|u(.,s)\|_{L^1(\Omega)}$. And since u is nonnegative, from [14, lemma 5.3] (valid for $q > 0$),

$$\int_\Omega u(t,.)dx + \int_s^t \int_\Omega |\nabla u|^q dx \leqq \int_\Omega u(s,.)dx. \tag{6.1}$$

Thus $\||\nabla u|^q\|_{L^1(Q_{\omega,s,\tau})}$ is bounded in terms of $\|u(.,s)\|_{L^1(\Omega)}$. Then one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}\left(\overline{\Omega} \times (0,T)\right)$ to a weak solution u of $(D_{\Omega,T})$. ∎

Remark 6.2 *In case of the Dirichlet problem, the result also follows from [7, Theorem 3.2 and Proposition 5.1], by using the uniqueness of the solution in $(Q_{\omega,\epsilon,T})$.*

Next we prove the uniqueness result of Theorem 1.6. For that purpose we recall a comparison property given in [1, Lemma 4.1]:

Lemma 6.3 ([1]) *Let Ω be bounded, and $A \in L^\sigma(Q_{\Omega,T})$ with $\sigma > N+2$. Let $w \in L^1((0,T);W^{1,1}_0(\Omega))$, with $w \in C((0,T];L^1(\Omega)$, such that $w_t - \Delta w \in L^1(Q_{\Omega,T})$, and $w(.,t)$ converges to a nonpositive measure $w_0 \in \mathcal{M}_b(\Omega)$, weakly in $\mathcal{M}_b(\Omega)$, and*

$$w_t - \Delta w \leqq A.\nabla w \qquad \text{in } \mathcal{D}'(Q_{\Omega,T}).$$

Then $w \leqq 0$ in $Q_{\Omega,T}$.

Proof of Theorem 1.6. From [7], the problems with initial data u_0, v_0 admit at least two solutions u, v. Then $f = |\nabla u|^q \in L^1_{loc}([0,T); L^1(\Omega))$. And by hypothesis $u \in C((0,T); L^1(\Omega)) \cap L^1((0,T); W_0^{1,1}(\Omega))$. Assume that $u_0 \leqq v_0$. Let $w = u - v$. Then we have $w \in C((0,T); L^1(\Omega)) \cap L^1((0,T); W_0^{1,1}(\Omega))$, $|\nabla w| \in L^k(Q_{\Omega,\tau})$ for any $k \in [1, q_*)$ and $\tau \in (0,T)$. Setting $g = |\nabla u|^q - |\nabla v|^q$, then w is the unique solution of the problem

$$\begin{cases} w_t - \Delta w = g, & \text{in } Q_{\Omega,T}, \\ w = 0, & \text{on } \partial\Omega \times (0,T), \\ \lim_{t\to 0} w(.,t) = u_0 - v_0, & \text{weakly in } \mathcal{M}_b(\Omega). \end{cases}$$

Since $q \leqq 1$, there holds

$$w_t - \Delta w = g \leqq |\nabla w|^q \leqq |\nabla w| + 1.$$

In case $q = 1$, Lemma 6.3 applies. Assume that $q < 1$. Let $\varepsilon, \eta \in (0,1)$. Then $g \leqq C_\eta |\nabla w| + \eta$. with $C_\eta = \eta^{-q/(1-q)}$. As in his proof we get by approximation

$$\begin{aligned} &\frac{1}{1+\varepsilon}\int_\Omega (w^+)^{1+\varepsilon}(t,.)dx + \varepsilon \int_0^t \int_\Omega (w^+)^{\varepsilon-1} |\nabla w|^2 \psi dx dt \\ &\leqq C_\eta \int_0^t \int_\Omega (w^+)^\varepsilon |\nabla w| \, dx dt + \eta \int_0^t \int_\Omega (w^+)^\varepsilon dx dt, \end{aligned}$$

and the second member is finite. Then $\lim_{t\to 0} \int_\Omega (w^+)^{1+\varepsilon}(t,.)dx = 0$, hence $\lim_{t\to 0} \int_\Omega w^+(t,.)dx = 0$. Let $z = w - \eta t$, then satisfies $z \in C((0,T); L^1(\Omega)) \cap L^1((0,T); W^{1,1}(\Omega))$ and $z_t - \Delta z = g - \eta \leqq C_\eta |\nabla z|$ in $\mathcal{D}'(Q_{\Omega,T})$. Then $z^+ \in C((0,T); L^1(\Omega)) \cap L^1((0,T); W_0^{1,1}(\Omega))$ and from [4, Lemma 3.2], $z_t^+ - \Delta z^+ \leqq C_\eta |\nabla(z^+)|$. And $\lim_{t\to 0} z^+(t) = 0$ weakly in $\mathcal{M}_b(\Omega)$, since $z^+ \leqq w^+$. Then $z^+ = 0$ from Lemma 6.3 applied with $A = C_\varepsilon$. Thus $w \leqq \eta t$; as $\eta \to 0$, we obtain $w \leqq 0$. ∎

Remark 6.4 *We can give an alternative proof of uniqueness, using regularity: let u, v be two solutions with initial data u_0, and $w = u - v$, thus w satisfies*

$$\begin{cases} w_t - \Delta w = g := |\nabla u|^q - |\nabla v|^q, & \text{in } Q_{\Omega,T}, \\ w = 0, & \text{on } \partial\Omega \times (0,T), \\ \lim_{t\to 0} w(.,t) = 0, & \text{weakly in } \mathcal{M}_b(\Omega). \end{cases} \tag{6.2}$$

Since $q \leqq 1$, there holds $|g| \leqq |\nabla w|^q$. As in Theorem 6.1, we choose $k \in (1, q_*)$, thus $|\nabla w| \in L^k(Q_{\Omega,\tau})$. From the uniqueness of the solution w due to [4, Lemma 3.4], we deduce that $w \in \mathcal{W}^{2,1,k}(Q_{\Omega,\tau})$, for any $\tau \in (0,T)$, from [22, theorem IV.9.1]. By induction we deduce that $w \in C^0\left(\overline{\Omega} \times [0,T)\right) \cap C^{2+\gamma,1+\gamma/2}(Q_{\Omega,T})$. Then $w = 0$ from the classical maximum principle.

Next we prove the trace result of Theorem 1.7:

First proof of Theorem 1.7. From Theorem 6.1, $u \in C^{2,1}_{loc}(Q_{\Omega,T})$. And $1 + u$ is also a solution of (1.1). We can set $1 + u = v^\alpha$, with $\alpha > 1$, in particular $v \geqq 1$. Then we obtain an equivalent equation for v :

$$v_t - \Delta v = H := (\alpha - 1)\frac{|\nabla v|^2}{v} - \alpha^{q-1}\frac{|\nabla v|^q}{v^{(\alpha-1)(1-q)}}.$$

From the Young inequality, setting $C = ((\alpha - 1)/2)^{(q-2)/q}$, there holds, since $v \geqq 1$,

$$\frac{|\nabla v|^q}{v^{(\alpha-1)(1-q)}} \leqq \frac{\alpha-1}{2}\frac{|\nabla v|^2}{v} + Cv^{1-\frac{2(1-q)}{2-q}\alpha} \leqq \frac{\alpha-1}{2}\frac{|\nabla v|^2}{v} + Cv.$$

Hence $w = e^{Ct} v$ satisfies

$$w_t - \Delta w = G := e^{Ct}(H + Cv) \geqq \frac{\alpha-1}{2}\frac{|\nabla w|^2}{w}.$$

Then w is supercaloric, and nonnegative, and $G \in L^1_{loc}(Q_T)$. From Lemma 3.1, w admits a trace in $\mathcal{M}(\Omega)$, and then $w \in L^\infty_{loc}([0,T); L^1_{loc}(\Omega))$, and $G \in L^1_{loc}([0,T); L^1_{loc}(\Omega))$. As a consequence, $v \in L^\infty_{loc}([0,T); L^1_{loc}(\Omega))$ and $|\nabla v|^2/v \in L^1_{loc}([0,T); L^1_{loc}(\Omega))$.

Next we show that moreover u itself admits a trace measure. For any $0 < s < t < T$, from the Hölder inequality,

$$\alpha^{-q}\int_s^t\int_\omega |\nabla u|^q\,dxdt = \int_s^t\int_\Omega v^{(\alpha-1)q}|\nabla v|^q\,dxdt \leqq \int_s^t\int_\omega \frac{|\nabla v|^2}{v}dxdt + \int_s^t\int_\omega v^{\frac{(2\alpha-1)q}{2-q}}dxdt. \tag{6.3}$$

First suppose $q < 1$. Choosing α such that moreover $1 < \alpha \leqq 1/q$, in order that $(2\alpha-1)q \leqq 2-q$. Since $v \in L^\infty_{loc}([0,T); L^1_{loc}(\Omega))$, we have $v \in L^1(Q_T)$, hence

$$\alpha^{-q}\int_s^t\int_\omega |\nabla u|^q\,dxdt \leqq \int_0^t\int_\omega \frac{|\nabla v|^2}{v}dxdt + \int_0^t\int_\omega (v+1)dxdt,$$

hence $|\nabla u|^q \in L^1_{loc}(\Omega\times[0,T))$. Then u admits a trace $u_0 \in \mathcal{M}^+(\Omega)$. Next assume $q = 1$. From the Hölder inequality,

$$\int_\omega |\nabla v|\,dx \leqq \int_\omega \frac{|\nabla v|^2}{v}dx + \int_\omega vdx$$

hence $|\nabla v| \in L^1_{loc}([0,T); L^1_{loc}(\Omega))$. Let $\xi \in \mathcal{D}(\Omega)$. Setting $v\xi = z$, z is the unique solution of the problem in $Q_{\Omega,T}$

$$\begin{cases} z_t - \Delta z = g := F\xi + v(-\Delta\psi) - 2\nabla v.\nabla\psi, & \text{in } Q_{\Omega,T},\\ z = 0, \quad \text{on } \partial\Omega\times(0,T), \\ \lim_{t\to 0} z(.,t) = \xi u_0, \ \text{weakly in } \mathcal{M}_b(\Omega), \end{cases}$$

where $g \in L^1(Q_{\Omega,T})$. From Theorem 2.11, for any $k \in [1, q_*)$, and for any $0 < s < \tau < T$, and any domain $\omega \subset\subset \Omega$,

$$\|z\|_{L^k(Q_{\omega,s,\tau})} \leqq C(\|F\xi\|_{L^1(Q_{\omega,s,\tau})} + \|z(s,.)\|_{L^1(\omega)}) \leqq C(\|F\xi\|_{L^1(Q_{\omega,\tau})} + \|v\|_{L^\infty((0,\tau);L^1(\omega))})$$

Then $z \in L^k_{loc}([0,T); L^k(\Omega))$. We can choose α such that $1 < \alpha < 1 + q_*/2$, and take $k = 2\alpha - 1$. From (6.3) we deduce that $|\nabla u| \in L^1_{loc}(\Omega\times[0,T))$, and conclude again that u admits a trace $u_0 \in \mathcal{M}^+(\Omega)$. ■

Finally we give an alternative proof by using comparison with solutions with initial Dirac mass, inspired of [2]. We first extend Proposition 2.14 to the case $q \leqq 1$ when Ω is bounded:

Lemma 6.5 *Let $0 < q \leqq 1$, Ω bounded, and $u_{0,n}, u_0 \in \mathcal{M}_b^+(\Omega)$ such that $u_{0,n}$ converge to u_0 weakly in $\mathcal{M}_b(\Omega)$. Let u_n, u be the unique nonnegative solutions of $(D_{\Omega,T})$ with initial data $u_{0,n}, u_0$. Then u_n converges to u in $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}\left(\overline{\Omega}\times(0,T)\right)$.*

Proof. We still have (2.12) and $\lim_{s\to 0}\int_\Omega u_n(s,.)dx = \int_\Omega du_{0,n}$, thus $\int_\Omega u_n(t,.)dx \leqq \int_\Omega du_{0,n}$, and $\lim_{n\to\infty}\int_\Omega du_{0,n} = \int_\Omega du_0$, thus (u_n) is bounded in $L^\infty((0,T); L^1(\Omega))$. From Theorem 6.1, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}\left(\overline{\Omega}\times(0,T)\right)$ to a weak solution w of $(D_{\Omega,T})$. And (u_n) is bounded in $L^k((0,T), W_0^{1,k}(\Omega)$ for any $k \in [1, q_*)$. As in Proposition 2.14, for any $\tau \in (0,T)$, $(|\nabla u_n|^q)$ is equi-integrable in $Q_{\Omega,\tau}$, and we conclude that $w = u$. ■

Second proof of Theorem 1.7. We still have $u \in C^{2,1}_{loc}(Q_{\Omega,T})$ from Theorem 6.1. It is enough to show that for any ball $B(x_0,\rho) \subset\subset \Omega$, there exists a measure $m_\rho \in \mathcal{M}(B(x_0,\rho))$ such that the restriction of u to $B(x_0,\rho)$ admits a trace $m_\rho \in \mathcal{M}(B(x_0,\rho))$. Suppose that it is not true. Then from Proposition 3.2 and Remark 3.3, there exists a ball $B(x_0,\rho) \subset\subset \Omega$ such that

$$\limsup_{t\to 0}\int_{B(x_0,\rho)} u(.,t)dx = \infty.$$

We can assume that $x_0 = 0$ and $\rho = 1$. For any $k > 0$, the Dirichlet problem $(P_{B_1,T})$ with initial data $k\delta_0$ has a unique solution $u_k^{B_1}$, from Theorem 1.6. As in the proof of Proposition 3.6, there exists $t_1 > 0$ such that $\int_{B_{2^{-1}}} u(x,t_1)dx > k$; thus there exists $s_{1,k} > 0$ such that $\int_{B_{2^{-1}}} \min(u(x,t_1), s_{1,k})dx = k$. By induction, there exists a decreasing sequence (t_n) converging to 0, and a sequence $(s_{n,k})$ such that $\int_{B_{2^{-n}}} \min(u(x,t_n), s_{n,k})dx = k$. Denote by $u_{n,k}$ the solution of $(P_{B_1,T})$ with initial data $u_{n,k,0} = \chi_{B_{2^{-n}}} \min(u(.,t_n), s_{n,k})$. Then $u \geqq u_{n,k}$ in B_1, from Theorem 1.6. And $(u_{n,k,0})$ converges weakly in $\mathcal{M}_b(\Omega)$ to $k\delta_0$. From Lemma 6.5, $(u_{n,k})$ converges in $C^{2,1}_{loc}(Q_{B_1,T}) \cap C^{1,0}\left(\overline{B_1} \times (0,T)\right)$ to the solution u^{k,B_1} of the problem in B_1 with initial data $k\delta_0$, Thus $u \geqq u^{k,B_1}$. Now, since $q \leqq 1$, for any $k > 1$, the function ku^{1,B_1} is a subsolution of (1.1), since $\left|\nabla(ku^{1,B_1})\right|^q \leqq k\left|\nabla(u^{1,B_1})\right|^q$. From Lemma 6.3, we deduce that $u \geqq ku^{1,B_1}$ for any $k > 1$. Since u^{1,B_1} is not identically 0, we get a contradiction as $k \to \infty$. ∎

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